Functional Analysis Exercise sheet 7 — solutions

1. Every $x \in H$ and written as $x = \sum_{n=1}^{\infty} x_n e_n$ with $\sum_{n=1}^{\infty} |x_n|^2 < \infty$. To compute the eigenvalues of A we need to solve $Ax = \lambda x$ which is equivalent to $x_{i+1} = \lambda x_i$ for $i \geq 1$. Then $x = \sum_{n=1}^{\infty} x_1 \lambda^{n-1} e_n$ is a general solution (when the series converges). If $|\lambda| < 1$, this defines an element in H, so that all λ with $|\lambda| < 1$ are eigenvalues. If $|\lambda| \geq 1$, the series diverges, so that such λ are not eigenvalues.

   Since for every $x \in H$, $\|Ax\| \leq \|x\|$, we deduce that $\|A\| \leq 1$. This implies that $\sigma(A) \subset \{|\lambda| \leq 1\}$. We also have $\{|\lambda| < 1\} \subset \sigma(A)$. Since $\sigma(A)$ is known to be closed, it follows that $\sigma(A) = \{|\lambda| \leq 1\}$.

2. Let $\mu = \lambda^n$ with $\lambda \in \sigma(A)$. Then the polynomial $x^n - \mu$ has a root λ, and $x^n - \mu = (x - \lambda)g(x)$ for another polynomial $g(x)$. Suppose that $A^n - \mu I$ has a bounded inverse B, namely,

$$(A^n - \mu I)B = B(A^n - \mu I) = I.$$

   We obtain

$$(A - \lambda I)g(A)B = Bg(A)(A - \lambda I) = I.$$

   Setting $C_1 = g(A)B$ and $C_2 = Bg(A)$, we get the bounded operators satisfying

$$(A - \lambda I)C_1 = C_2(A - \lambda I) = I.$$

   Moreover,

$$C_1 = C_2(A - \lambda I)C_1 = C_2.$$

   This proves that $A - \lambda I$ is invertible, but $\lambda \in \sigma(A)$. Hence, $A^n - \mu I$ cannot be invertible not invertible. This proves that

$$\{\lambda^n : \lambda \in \sigma(A)\} \subset \sigma(A^n).$$

   Let $\mu \in \sigma(A^n)$, i.e., $A^n - \mu I$ is not invertible. We have

$$A^n - \mu I = \prod_{i=1}^{n} (A - \lambda_i I),$$

   where λ_i's are the roots of $x^n - \mu = 0$. If all $\lambda_i \notin \sigma(A)$, then the operators $A - \lambda_i I$ are invertible, and $A^n - \mu I$ would be also invertible. Hence, we conclude that for some i, $A - \lambda_i I$ is not invertible, and $\lambda_i \in \sigma(A)$. This proves that

$$\sigma(A^n) \subset \{\lambda^n : \lambda \in \sigma(A)\}.$$

3. We know that $\ell^p \subsetneq \ell^2$ for $p < 2$. We take $x^0 \in \ell^2 \backslash \ell^p$. Suppose that for some $y^0 \in \ell^p$ and $\epsilon > 0$,

$$\{y \in \ell^2 : \|y - y^0\|_2 < \epsilon\} \subset \ell^p.$$

Then $z = y^0 + \frac{\epsilon}{2\|x^0\|_2} x^0$ belongs to this set. However, then $x^0 = \frac{2\|x^0\|_2}{\epsilon}(z - y^0)$ would belong to ℓ^p. This gives a contradiction. Hence, we conclude that ℓ^p has empty interior in ℓ^2.

We claim that the sets

$$B_R = \{x \in \ell^2 : \|x\|_p \le R\}$$

are closed in ℓ^2. Suppose that $x^{(n)} = (x_k^{(n)})_{k \ge 1} \in B_R$ and $x^{(n)} \to x$ in ℓ^2. This implies that for every k, $x_k^{(n)} \to x_k$ as $n \to \infty$. Hence, for every $N \ge 1$,

$$\sum_{k=1}^{N} |x_k|^p = \sum_{k=1}^{N} \lim_{n \to \infty} |x_k^{(n)}| = \lim_{n \to \infty} \sum_{k=1}^{N} |x_k^{(n)}|^p \le R^p,$$

and

$$\|x\|_p^p = \lim_{N \to \infty} \sum_{k=1}^{N} |x_k|^p \le R^p.$$

This proves the claim.

Since $\ell^p = \cup_{m \ge 1} B_m$ where B_m's are closed and have empty interior, ℓ^p is meager in ℓ^2.

We observe that $\cup_{p<2} \ell^p = \cup_{n \ge 1} \ell^{2-1/n}$. Since the sets $\ell^{2-1/n}$ are meager, it follows that the union is meager too. By the Baire Category Theorem, $\cup_{p<2} \ell^p \ne \ell^2$. This implies (b).

4. Since B is linear, it is continuous if and only if B is continuous at 0. It is sufficient to show that for $x_n \to 0$ and $y_n \to 0$, we have $B(x_n, y_n) \to 0$. Let $T_n(y) = B(x_n, y)$. According to our assumption the map T_n is continuous, so that it is bounded. For fixed y, the linear map $x \mapsto B(x, y)$ is bounded. Hence, it follows that the sequence $T_n(y)$ is bounded. Applying the Uniform Boundedness Principle, we deduce that there exists $C > 0$ such that $\|T_n\| \le C$ for all n. Thus,

$$|B(x_n, y_n)| = |T_n(y_n)| \le C\|y_n\| \to 0.$$

This completes the proof.

Let $f(x, y) = \frac{xy}{x^2+y^2}$ for $(x, y) \ne (0, 0)$ and $f(0, 0) = 0$. The functions $f(x, \cdot)$ and $f(\cdot, y)$ are continuous. However, the function f is not continous at $(0, 0)$ because $f(1/n, 1/n) = 1/2$.

5. Suppose that $\|A\| = \infty$. This means that there exists $x_n \in H$ such that $\|y_n\| = 1$ and $\|Ay_n\| \to \infty$. We consider the sequence of maps $f_n(x) = \langle x, Ay_n \rangle$, $x \in H$. They define bounded linear functionals on H with $\|f_n\| = \|Ay_n\| \to \infty$. On the other hand, for every $x \in H$,

$$|f_n(x)| = |\langle x, Ay_n \rangle| = |\langle Ax, y_n \rangle| \leq \|Ax\|\|y_n\| = \|Ax\|.$$

Hence, it follows from the Uniform Boundedness Principle that the sequence of norms $\|f_n\|$ is bounded. This contradiction implies that $\|A\| < \infty$.