Functional Analysis Exercise sheet 8

1. We say that a sequence x_n in a normed space X is a weak Cauchy sequence if for every f in X^* the sequence $f(x_n)$ is Cauchy. Show that a weak Cauchy sequence is bounded.

2. Let H be a Hilbert space.

   (a) Show that $x_n \to x$ in norm if and only if $x_n \to x$ weakly and $\|x_n\| \to \|x\|$.

   (b) Suppose that $x_n \to x$ in norm and $y_n \to y$ weakly. Show that $\langle x_n, y_n \rangle \to \langle x, y \rangle$.

   (c) Suppose that $x_n \to x$ weakly and $y_n \to y$ weakly. Is it true that $\langle x_n, y_n \rangle \to \langle x, y \rangle$? Justify your answer.

3. Let $X = C([0,1])$ equipped with the max-norm. Consider a sequence of linear functionals

$$L_n(f) = n \int_0^{1/n} f(t)\, dt, \qquad f \in C([0,1]),$$

   on X.

   (a) Show that the sequence L_n weak* converges.

   (b) Does the sequence L_n converge in norm? Justify your answer.

4. (a) Let $(e_k)_{k \geq 1}$ be a complete orthonormal set of a Hilbert space H. We consider a sequence of operators P_n defined by

$$P_n x = \sum_{k=1}^{n} \langle x, e_k \rangle\, e_k \quad \text{for } x \in H.$$

   Prove that $\|P_n - P_m\| = 1$ for $n \neq m$ and hence P_n does not converge in norm.

   (b) Prove that P_n defined in (a) converges to the identity operator strongly.

5. Let X be a Banach space and $A_n, A, B_n, B \in B(X, X)$.

   (a) Show that if $A_n \xrightarrow{s} A$ and $B_n \xrightarrow{s} B$, then $A_n B_n \xrightarrow{s} AB$.

   (b) Is it true that if $A_n \xrightarrow{w} A$ and $B_n \xrightarrow{w} B$, then $A_n B_n \xrightarrow{w} AB$? Justify your answer.