Functional Analysis Exercise sheet 8 — solutions

1. Each element x_n defines a linear map $L_n : X^* \to \mathbb{C}$ by $L_n(f) = f(x_n)$. Since the sequence $f(x_n)$ is Cauchy, it is bounded. Hence, there exists $c = c(f) > 0$ such that $|L_n(f)| \le c$ for all n. We note that X^* is also a Banach space. By the Uniform Boundedness Theorem, $\sup_n \|L_n\| < \infty$. Finally,

$$\|L_n\| = \sup\{|L_n(f)| : \|f\| = 1\} = \sup\{|f(x_n)| : \|f\| = 1\} = \|x_n\|.$$

This proves that the sequence x_n is bounded.

2. (a) Suppose that $\|x_n - x\| \to 0$. Then for every $y \in H$,

$$|\langle x_n, y\rangle - \langle x, y\rangle| \le \|x_n - x\|\|y\| \to 0.$$

Hence, $x_n \to x$ weakly. Also by triangle inequality,

$$|\|x_n\| - \|x\|| \le \|x_n - x\| \to 0.$$

To prove the converse, we observe that

$$\|x - x_n\|^2 = \|x\|^2 - \langle x_n, x\rangle - \overline{\langle x_n, x\rangle} + \|x_n\|^2.$$

By weak convergence, $\langle x_n, x\rangle \to \|x\|^2$. Hence, $\|x - x_n\|^2 \to 0$ as required.

(b) We have

$$\begin{aligned} |\langle x_n, y_n\rangle - \langle x, y\rangle| &\le |\langle x_n, y_n\rangle - \langle x, y_n\rangle| + |\langle x, y_n\rangle - \langle x, y\rangle| \\ &= |\langle x_n - x, y_n\rangle| + |\langle x, y_n - y\rangle|. \end{aligned}$$

Since $y_n \to y$ weakly, the second term converges to zero. By Cauchy-Schwarz inequality,

$$|\langle x_n - x, y_n\rangle| \le \|x_n - x\|\|y_n\|.$$

Since y_n is weakly convergent, it is bounded. Hence, it follows that $|\langle x_n - x, y_n\rangle| \to 0$.

(c) This is not true in general. For example, consider $H = \ell^2$ and $x_n = y_n = e_n$. Then $e_n \to 0$ weakly, but $\langle e_n, e_n\rangle = 1$ does not converge to zero.

3. (a) Let $L(f) = f(0)$. We claim that $L_n \to L$ weak*. Since f is continuous, for every $\epsilon > 0$ and $n \ge n_0(\epsilon)$, we have $|f(t) - f(0)| < \epsilon$ for all $t \in [0, 1/n]$. Then

$$|L_n(f) - L(f)| \le n \int_0^{1/n} |f(t) - f(0)|\, dt < \epsilon.$$

Hence, $L_n(f) \to L(f)$ for all $f \in C([0,1])$.

(b) It follows form (a) that if $L_n \to S$ in norm, then $S = L$. Consider a function f_n such that $0 \leq f_n \leq 1$, $f_n(0) = 0$, and $f_n = 1$ on $[\frac{1}{3n}, \frac{2}{3n}]$. Then $\|f_n\| = 1$, and

$$|L_n(f_n) - L(f_n)| = n \left| \int_0^{1/n} f_n(t)\, dt \right| \geq \frac{1}{3}.$$

Hence, $\|L_n - L\| \geq \frac{1}{3}$, and the sequence L_n does not converge in norm.

4. (a) For $n < m$,

$$\|P_n x - P_m x\|^2 = \left\| \sum_{k=n+1}^{m} \langle x, e_k \rangle e_k \right\|^2 = \sum_{k=n+1}^{m} | \langle x, e_k \rangle |^2.$$

Hence, by the Bessel inequality, $\|P_n x - P_m x\|^2 \leq \|x\|^2$. This proves that $\|P_n - P_m\| \leq 1$. For $x = e_m$, $\|P_n x - P_m x\| = \|e_m\|$. Hence, $\|P_n - P_m\| = 1$.

If we suppose $P_n \to P$ for some P in norm, then $\|P_n - P_m\| \leq \|P_n - P\| + \|P - P_m\| \to 0$ as $n, m \to \infty$, which is impossible.

(b) We recall that for every $x \in H$, $x = \sum_{k=1}^{\infty} \langle x, e_k \rangle e_k$ and $\sum_{k=1}^{\infty} | \langle x, e_k \rangle |^2 < \infty$. Hence,

$$\begin{aligned} \|P_n x - I x\| &= \left\| \sum_{k=n+1}^{\infty} \langle x, e_k \rangle e_k \right\| = \lim_{N \to \infty} \left\| \sum_{k=n+1}^{N} \langle x, e_k \rangle e_k \right\| \\ &= \lim_{N \to \infty} \sqrt{\sum_{k=n+1}^{N} | \langle x, e_k \rangle |^2} = \sqrt{\sum_{k=n+1}^{\infty} | \langle x, e_k \rangle |^2}, \end{aligned}$$

and $\|P_n x - I x\| \to 0$ as $n \to \infty$.

5. (a) We first note that since A_n strongly converge, the sequence of norms $\|A_n\|$ is bounded. We fix $c > 0$ such that $\|A_n\| \leq c$ for all n. For every $x \in X$,

$$\begin{aligned} \|A_n B_n x - ABx\| &\leq \|A_n B_n x - A_n B x\| + \|A_n B x - ABx\| \\ &\leq \|A_n\| \|B_n x - Bx\| + \|A_n(Bx) - A(Bx)\| \\ &\leq c\|B_n x - Bx\| + \|A_n(Bx) - A(Bx)\| \end{aligned}$$

Since $A_n \xrightarrow{s} A$ and $B_n \xrightarrow{s} B$, we have $\|B_n x - Bx\| \to 0$ and $\|A_n(Bx) - A(Bx)\| \to 0$. Hence,

$$\|A_n B_n x - ABx\| \to 0.$$

(b) Consider the operators $A_n, B_n : \ell^2 \to \ell^2$ defined by $B_n x = \langle x, e_1 \rangle e_n$ and $A_n x = \langle x, e_n \rangle e_1$. For every $x, y \in \ell^2$, $\langle B_n x, y \rangle = \langle x, e_1 \rangle \langle e_n, y \rangle \to 0$, and $\|A_n x\| \to 0$. Hence, $A_n \xrightarrow{s} A$ and $B_n \xrightarrow{w} B$. We also have $A_n B_n x = \langle x, e_1 \rangle e_1$. Hence, $A_n B_n \overset{w}{\not\to} AB$.