Functional Analysis Exercise sheet 9

1. Let X be the normed space which consists of sequences of complex numbers with only finitely many nonzero terms where $\|\xi\| = \sup_{j\in\mathbb{N}} |\xi_j|$. Let $T : X \to X$ be defined by
$$T(x_1, x_2, x_3, \ldots) = (x_1, x_2/2, x_3/3, \ldots).$$
Show that T is bounded and bijective but that T^{-1} is unbounded. Why does this not contradict the Open Mapping Theorem?

2. Let X and Y be normed spaces and $T : X \to Y$ a closed linear operator. Show that
$$\ker(T) = \{x \in X : T(x) = 0\}$$
is a closed subspace of X.

3. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ are norms on vector space X such that $(X, \|\cdot\|_1)$ and $(X, \|\cdot\|_2)$ are complete. Suppose that these norms have the property that for every sequence x_n,
$$\|x_n - y_1\|_1 \to 0 \text{ and } \|x_n - y_2\|_2 \to 0 \ \Rightarrow\ y_1 = y_2.$$
Prove that there exists $c_1, c_2 > 0$ such that
$$c_1\|x\|_1 \le \|x\|_2 \le c_2\|x\|_1 \quad \text{for all } x \in X.$$

4. Let X and Y be Banach spaces, and $T : \mathcal{D} \to Y$ a linear operator defined on a subspace $\mathcal{D}$ of X. An operator $\hat{T} : \hat{\mathcal{D}} \to Y$ is called an extension of the operator T if $\mathcal{D} \subset \hat{\mathcal{D}}$ and $\hat{T} = T$ on $\mathcal{D}$.

   Show that T has an extension $\hat{T} : \hat{\mathcal{D}} \to Y$ which is a closed linear operator if and only if the closure of the graph $\overline{\Gamma(T)}$ does not contain an element of the form $(0, y)$ with $y \neq 0$.