Functional Analysis Exercise sheet 9

1. Since for every $x \in X$,

$$\|Tx\| = \sup_{j\geq 1} j^{-1}|x_j| \leq \|x\|,$$

The inverse of T is computed as

$$T^{-1}(x_1, x_2, x_3, \ldots) = (x_1, 2x_2, 3x_3, \ldots).$$

Since $\|T^{-1}e_n\| = n$ with $\|e_n\| = 1$, it follows that $\|T^{-1}\| = \infty$. This does not contradict the Open Mapping Theorem because the space X is not complete.

2. Since T is closed, we know that for every sequences $x_n \in X$ and $y_n \in Y$ such that $Tx_n = y_n$, if $x_n \to x$ and $y_n \to y$, then $Tx = y$. Take any x in the closure of $\mathrm{Ker}(T)$. Then there exists $x_n \in \mathrm{Ker}(T)$ such that $Tx_n = 0$. Hence, it follows that $Tx = 0$, so that $x \in \mathrm{Ker}(T)$. This proves that the kernel is closed.

3. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ are norms on vector space X such that $(X, \|\cdot\|_1)$ and $(X, \|\cdot\|_2)$ are complete. Suppose that these norms have the property that for every sequence x_n,

$$\|x_n - x_1\|_1 \to 0 \text{ and } \|x_n - x_2\|_2 \to 0 \;\Rightarrow\; x_1 = x_2.$$

Prove that there exists $c_1, c_2 > 0$ such that

$$c_1\|x\|_1 \leq \|x\|_2 \leq c_2\|x\|_1 \quad \text{for all } x \in X.$$

We consider the identity map $I : (X, \|\cdot\|_1) \to (X, \|\cdot\|_2)$. The graph of I is $\Gamma(I) = \{(x, x) : x \in X\}$. It follows from our assumption that if $(x_n, x_n) \to (y_1, y_2)$, then $y_1 = y_2$. This implies that $\Gamma(I)$ is closed. Hence, by the Closed Graph Theorem, I is bounded. Moreover, by the Open Mapping Theorem, I^{-1} is bounded too. This implies the result.

4. Suppose that T has an extension $\hat{T}$ which is a closed linear operator. Then $\Gamma(T) \subset \Gamma(\hat{T})$ and $\overline{\Gamma(T)} \subset \Gamma(\hat{T})$. Since $\Gamma(\hat{T}) = \{(x, \hat{T}x) : x \in \hat{\mathcal{D}}\}$, it is clear that if $(0, y) \in \Gamma(\hat{T})$, then $y = 0$. This proves one of the implications.

To prove the other implication, we show that the closure $\overline{\Gamma(T)}$ defines a graph of a linear operator. Since T is linear, it follows that $\Gamma(T)$ and $\overline{\Gamma(T)}$ are linear subspaces in $X \times Y$. Let $\hat{\mathcal{D}}$ be the projection of $\overline{\Gamma(T)}$ to X. If $(x, y_1), (x, y_2) \in \overline{\Gamma(T)}$, then also

$$(x, y_1) - (x, y_2) = (0, y_1 - y_2) \in \overline{\Gamma(T)}.$$

So that according to our assumption, $y_1 = y_2$. This proves for every $x \in \hat{\mathcal{D}}$, there exists a unique $y \in Y$ such that $(x, y) \in \overline{\Gamma(T)}$. Hence, $\overline{\Gamma(T)}$ defines a map $\hat{T} : \hat{\mathcal{D}} \to Y$ with the graph $\overline{\Gamma(T)}$. Since $\overline{\Gamma(T)}$ is a subspace, it is clear that $\hat{T}$ is linear. Since $\overline{\Gamma(T)}$ is closed, $\hat{T}$ is closed.