EXAMINATION SOLUTIONS

FUNCTIONAL ANALYSIS
MATH 36202
(Paper Code MATH M6202)

May-June 2015, 2 hours and 30 minutes

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1. (a) **(5 marks, bookwork)**
Using the linearity of the inner product,

$$\|x+ty\|^2 = \langle x+ty, x+ty\rangle = \|x\|^2 + 2t\langle x,y\rangle + t^2\|y\|^2 \geq 0.$$

Since this quadratic polynomial is always non-negative, its discriminant satisfies $4\langle x,y\rangle^2 - 4\|x\|^2\|y\|^2 \leq 0$. Hence, $|\langle x,y\rangle| \leq \|x\|\,\|y\|$, as required.

(b) **(5 marks, similar to homework)**
By the Cauchy-Schwarz inequality,

$$|\langle x_n, y_n\rangle - \langle x,y\rangle| \leq |\langle x_n, y_n - y\rangle| + |\langle x_n - x, y\rangle| \leq \|x_n\|\|y_n - y\| + \|x_n - x\|\|y\|.$$

We observe that $\|x_n\| \leq \|x\| + \|x_n - x\| \to \|x\|$, so that $\|x_n\|$ is bounded. Since $\|x_n - x\| \to 0$ and $\|y_n - y\| \to 0$, this implies that the above expression converges to zero.

(c) **(5 marks, unseen)**
Observe that

$$\|tx + (1-t)y\| \leq \|tx\| + \|(1-t)y\| = 1,$$

for any $0 < t < 1$.
Suppose in contrary that the equality holds for some t in $(0,1)$. Then

$$1 = \langle tx + (1-t)y, tx + (1-t)y\rangle = t^2 + 2t(1-t)\langle x,y\rangle + (1-t)^2,$$

and we obtain $\langle x,y\rangle = 1 = \|x\|\|y\|$ — the equality case in the Cauchy–Schwarz inequality. This is only possible when the vectors x and y are linearly dependent, which gives a contradiction.

(d) (i) **(2 marks, similar to homework)**
If $u, v \in H_0$ and $a, b \in \mathbb{F}$, then $\langle au + bv, x_0\rangle = a\langle u, x_0\rangle + b\langle v, x_0\rangle = 0$. Hence, H_0 is a subspace.
Suppose that $x_n \in H_0$ and $x_n \to x \in H$. Then

$$|\langle x_n, x_0\rangle - \langle x_n, x_0\rangle| = |\langle x_n - x, x_0\rangle| \leq \|x_n - x\|\|x_0\| \to 0.$$

This implies that $\langle x, x_0\rangle = 0$ as required.

(ii) **(3 marks, similar to bookwork)**
Given $y \in H$, we write $y = y_0 + cx_0$ where $c = \frac{\langle y, x_0 \rangle}{\|x_0\|^2}$ and $y_0 = y - cx_0$. Then for $z \in H_0$, we have $y - z = (y_0 - z) + cx_0$ where $(y_0 - z) \perp cx_0$. Hence

$$\|y - z\|^2 = \|y_0 - z\|^2 + |c|^2 \|x_0\|^2.$$

This shows that $\|y - z\| \geq |c| \|x_0\| = \frac{|\langle y, x_0 \rangle|}{\|x_0\|}$, where equality holds when $y = z$. Thus, the distance from y to H_0 is $\frac{|\langle y, x_0 \rangle|}{\|x_0\|}$.

(e) (i) **(2 marks, bookwork)**
Let (e_n) be an orthonormal system in H. Then for every $x \in H$,

$$\sum_n |\langle x, e_n \rangle|^2 \leq \|x\|^2.$$

(ii) **(3 marks, unseen)**
Suppose that the Bessel inequality holds. Taking $x = e_1$, we deduce that

$$|\langle e_1, e_1 \rangle|^2 + \sum_{n \geq 2} |\langle e_1, e_n \rangle|^2 \leq \|e_1\|^2$$

and this implies that $\langle e_1, e_n \rangle = 0$ for all $n \geq 2$. The same argument shows that $\langle e_k, e_l \rangle = 0$ for all $k \neq l$.

2. (a) **(4 marks, bookwork)**
Let $x^{(n)} = (x_k^{(n)})_{k \geq 1}$ be a Cauchy sequence in ℓ^∞. This means that for every $\epsilon > 0$, and $n, m \geq n_0(\epsilon)$,

$$\|x^{(n)} - x^{(m)}\|_\infty = \sup_k |x_k^{(n)} - x_k^{(m)}| < \epsilon.$$

This, in particular, implies that for every k and $n, m \geq n_0(\epsilon)$,

$$|x_k^{(n)} - x_k^{(m)}| < \epsilon.$$

In particular, each sequence $(x_k^{(n)})_{n \geq 1}$ is a Cauchy sequence. Since $\mathbb{R}$ is complete, $x_k^{(n)} \to x_k$ as $n \to \infty$ for some $x_k \in \mathbb{R}$. Passing to the limit as $m \to \infty$ in the above inequality, we deduce that for every k and $n \geq n_0(\epsilon)$,

$$|x_k^{(n)} - x_k| \leq \epsilon.$$

This implies that for every k, $|x_k| < \epsilon + \|x^{(n)}\|_\infty$, and $x = (x_k)_{k \geq 1}$ belongs to ℓ^∞. The above inequality also implies that $\|x^{(n)} - x\|_\infty \to 0$ as $n \to \infty$.

(b) **(5 marks, similar to bookwork)**
Let S be complete subspace of a Banach space X. Let $x_n \in S$ such that $x_n \to x \in X$. Then $\|x_n - x_m\| \leq \|x_n - x\| + \|x_m - x\| \to 0$ as $n, m \to \infty$, so that the sequence x_n is Cauchy. Hence, $x_n \to y$ for some $y \in S$. Since the limit is unique $x = y \in S$. This shows that S is closed.

Now suppose that S is a closed subspace of X, and x_n is a Cauchy sequence in S. Since X in Banach space, by completeness $x_n \to x$ for some $x \in X$, but since S is closed, $x \in S$. This proves that x_n converges in S, and S is complete.

(c) (i) **(4 marks, similar to homework)**
We consider $x = (1/k)_{k \geq 1} \in \ell^\infty$ and $x^{(n)} \in X$ such that $x_k^{(n)} = 1/k$ for $k \leq n$ and $x_k^{(n)} = 0$ for $k > n$. Then $\|x^{(n)} - x\| \leq 1/(n+1) \to 0$. This shows that x belongs to the closure of X, and X is not closed (hence, not complete) in ℓ^∞.

(ii) **(5 marks, similar to homework)**
We claim that $\bar{X}$ consists of sequences $x = (x_k)_{k\geq 1}$ such that $x_k \to 0$. Let x be such a sequence. Then for every $\epsilon > 0$ and $n \geq n_0(\epsilon)$, we have $|x_k| < \epsilon$. We take $x_k^{(n)} = x_k$ for $k \leq n_0(\epsilon)$ and $x_k^{(n)} = 0$ for $k > n$. Then $\|x^{(n)} - x\|_\infty < \epsilon$. This proves that $x \in \bar{X}$. Conversely, suppose that $x^{(n)} \to x$ for some $x^{(n)} \in X$. Then for every $\epsilon > 0$ and sufficiently large n, $\|x^{(n)} - x\|_\infty < \epsilon$. Since $x^{(n)} \in X$, for all sufficiently large k, $|x_k^{(n)} - x_k| = |x_k|$. This implies that for all sufficiently large k, $|x_k| < \epsilon$. Hence, $x_k \to 0$, as claimed.

(iii) **(4 marks, similar to homework)**
Consider the operator $S : X \to X$ defined by $(x_n) \to (2^n x_n)$. Then $TS = ST = I$. Take $x^{(k)} \in X$ such that $x_n^{(k)} = 1$ for $n \leq k$ and $x_n^{(k)} = 0$ for $n > k$. Then $\|x^{(k)}\|_\infty = 1$ and $\|Sx^{(k)}\|_\infty = 2^k$. Since $\|S\| = \sup\{\|Sx\|_\infty : \|x\|_\infty = 1\}$, this shows that $\|S\| = \infty$.

(d) **(3 marks, bookwork)**
The Bounded Inverse Theorem says that if $T : X \to X$ is a bounded bijective linear map where X is a Banach space, then the inverse map T^{-1} is also bounded. This theorem does not apply to (c)(iii) because X in (c) is not not complete as shown in (c)(i) so it is not a Banach space.

3. (a) **(2 marks, bookwork)**
The dual space X^* is the space of bounded linear maps $f : X \to \mathbb{C}$.

(b) **(4 marks, bookwork)**
We claim that $(\ell^1)^* \simeq \ell^\infty$. Let $a = (a_k)_{k\geq 1} \in \ell^\infty$. We defined $f_a : \ell^1 \to \mathbb{C}$ by $f_a(x) = \sum_{k\geq 1} a_k x_k$. It is easy to check that f_a is linear. Also

$$|f_a(x)| \leq \sum_{k\geq 1} |a_k x_k| \leq \|a\|_\infty \|x\|_1.$$

Hence, $f_a \in X^*$. Now take any $f \in X^*$. We denote by $e_k \in \ell^1$ the vector whose k's coordinate is 1 and the other coordinates are 0. Let $a_k = f(e_k)$. Then $|a_k| \leq \|f\|\|e_k\|_1 = \|f\|$, so that $a = (a_k)_{k\geq 1} \in \ell^\infty$. By linearity $f(x) = f_a(x)$ for all x in the subspace X of ℓ^1 consisting of x that have only finitely many non-zero coordinates. This subspace is dense in ℓ^1. Hence, by continuity $f(x) = f_a(x)$ for all $x \in \ell^\infty$.

(c) **(3 marks, similar to homework)**
Since $\sum_{n\geq 1} \|x_n\| < \infty$, for every $\epsilon > 0$ and $m \geq m_0(\epsilon)$, $\sum_{n\geq m} \|x_n\| < \epsilon$. Consider the sequence $s_m = \sum_{n=1}^m x_n$. For $m_1 < m_2$ we have $\|s_{m_1} - s_{m_2}\| \leq \sum_{n=m_1+1}^{m_2} \|x_n\|$. Hence, if $m_1 \geq m_0(\epsilon)$, then $\|s_{m_1} - s_{m_2}\| < \epsilon$. This shows that the sequence s_m is Cauchy. Since X is a Banach space, this sequence must converge.

(d) **(3 marks, similar to homework)**
We apply the Hahn-Banach theorem. Let V be the subspace of X spanned by x. We define $f \in X^*$ by $f(ax) = a$. Note that $|f(ax)| = |a| = \|ax\|$. By the Hahn-Banach theorem f can be expended to a linear map $X \to \mathbb{C}$ such that $|f(x)| \leq \|x\|$ for all $x \in X$. In particular, $|f| \leq 1$ on B.

(e) **(5 marks, unseen)**
Each element x_n defines a linear map $L_n : X^* \to \mathbb{C}$ by $L_n(f) = f(x_n)$. Since the sequence $f(x_n)$ is Cauchy, it is bounded. Hence, there exists $c = c(f) > 0$ such that $|L_n(f)| \leq c$ for all n. We note that X^* is also a Banach space. By the Uniform Boundedness Theorem, $\sup_n \|L_n\| < \infty$. Finally,

$$\|L_n\| = \sup\{|L_n(f)| : \|f\| = 1\} = \sup\{|f(x_n)| : \|f\| = 1\} = \|x_n\|.$$

This proves that the sequence x_n is bounded.

(f) (i) **(3 marks, bookwork)**
$A_n \to A$ in norm topology if $\|A_n - A\| \to 0$. $A_n \to A$ in strong topology if $\|A_n x - Ax\| \to 0$ for every $x \in X$. $A_n \to A$ in weak topology if $f(A_n x - Ax) \to 0$ for every $x \in X$ and $f \in X^*$.

(ii) **(5 marks, unseen)**
For $k > n$, $S^n e_k = e_{n-k}$, so that for $k > n_2 > n_1$,

$$\|(S^{n_1} - S^{n_2})e_k\|_1 = \|e_{k-n_1} - e_{k-n_2}\|_1 = 2.$$

Hence, $\|S^{n_1} - S^{n_2}\| \geq 2$. This shows that S^n is not a Cauchy sequence, and it cannot converge in the norm topology.
Take $x \in \ell^1$. For every $\epsilon > 0$, there exists x_0 that has only finitely many nonzero coordinates such that $\|x - x_0\| < \epsilon$. Then for all sufficiently large n, $S^n x_0 = 0$, and $\|S^n x\|_1 \leq \|S^n x_0\|_1 + \|S^n\| \|x - x_0\|_1 \leq \|x - x_0\|_1 < \epsilon$. This proves that $S^n \to 0$ in strong topology.
Alternatively, to prove that S^n does not converge in norm topology, one notes that $\|S^n\| = 1$. If $S^n \to T$ for some operator T, then $\|T\| = 1$, but $T = 0$ because $S^n \to 0$ in weak topology. This gives a contradiction.

4. (a) **(2 marks, bookwork)**
Every bounded linear map $f : H \to \mathbb{C}$ is of the form $f(x) = \langle x, y \rangle$ for a uniquely defined $y \in H$. Moreover, $\|f\| = \|y\|$.

(b) **(2 marks, bookwork)**
$x_n \to x$ weakly if $\langle x_n - x, z \rangle \to 0$ for every $z \in H$.

(c) **(3 marks, unseen)**
$\langle x_n - y, w \rangle \to 0$ and $\langle x_n - z, w \rangle \to 0$ for every $w \in H$. Then by subtracting we obtain $\langle y - z, w \rangle = 0$. Since this holds for all $w \in H$, this implies that $y = z$.

(d) (i) **(4 marks, unseen)**
We need to show that for every $f \in H^*$, $f(Ax_n - Ax) \to 0$. The map $g(x) = f(Ax)$ is a bounded linear map. Hence, $g(x_n - x) \to 0$ as required.

(ii) **(5 marks, unseen)**
Suppose that the operator A is unbounded. Then there a sequence of unit vectors x_n such $\|Ax_n\| \geq n^2$. Let $y_n = x_n/n$. Then $\|y_n\| \to 0$ and $\|Ay_n\| \to \infty$. By the assumption $Ay_n \to 0$ weakly. In particular, the sequence $\langle Ay_n, y \rangle$ is bounded for every $y \in H$. Let $L_n(y) = \langle Ay_n, y \rangle$. Note that $L_n \in H^*$ and $\|L_n\| = \|Ay_n\|$. By the Uniform Boundedness Principle, $\sup_n \|Ay_n\| < \infty$. This gives a contradiction.

(e) **(4 marks, homework)**
Recall that $(ST)^* = T^* S^*$. We have $AA^{-1} = A^{-1}A = I$ and hence $(A^{-1})^* A^* = A^* (A^{-1})^* = I$. This proves that $(A^{-1})^*$ is the inverse of A^*. Since $\|(A^{-1})^*\| \leq \|A^{-1}\|$, the inverse is bounded.

(f) **(5 marks, homework)**
For $f \in H$, $\|A_\phi f\|_2 \leq \|\phi\|_\infty \|f\|_2$. Hence, $\|A_\phi\|_2 \leq \|\phi\|_\infty$. We claim that the equality holds. Let $m = \|\phi\|_\infty$. Since ϕ is continuous, there exists $x_0 \in [0,1]$ such $|\phi(x_0)| = m$. For every $\epsilon > 0$, there exists $\delta > 0$ such that $|\phi(x_0) - \phi(x)| < \epsilon$ for all $x \in B_\delta(x_0)$. We take $f \in H$ such that $\mathrm{supp}(f) \subset B_\delta(x_0)$. Then

$$\|A_\phi f\|_2 \geq \|A_m f\|_2 - \|(A_\phi - A_m)f\|_2 \geq m\|f\|_2 - \epsilon\|f\|_2.$$

This shows that $\|A_\phi\|_2 \geq m - \epsilon$ for every $\epsilon > 0$. Hence, $\|A_\phi\|_2 = m$.
For $f_1, f_2 \in H$, $\langle A_\phi f_1, f_2 \rangle = \langle \phi f_1, f_2 \rangle = \langle f_1, \bar{\phi} f_2 \rangle = \langle f_1, A_{\bar{\phi}} f_2 \rangle$. Hence, $A_\phi^* = A_{\bar{\phi}}$.

5. (a) **(3 marks, bookwork)**
A subset is meager if it is a countable union of sets whose closure has empty interior. The Baire Category Theorem states that a complete metric space is not meager.

(b) **(3 marks, similar to homework)**
Consider the linear maps $L_n(f) = \int_0^1 \phi_n(x) f(x) dx$. We claim that $\|L_n\| = \|\phi_n\|_2$. Indeed, $|L_n(f)| \leq \|\phi_n\|_2 \|f\|_2$, so that $\|L_n\| \leq \|\phi_n\|_2$. Taking $f = \bar{\phi}_n$, $|L_n(f)| = \|\phi\|_2 \|f\|_2$, so that we deduce that $\|L_n\| \geq \|\phi_n\|_2$ which proves the claim. The space $L^2([0,1])$ is complete, and we can apply the Uniform Boundedness Theorem to the linear maps. Suppose that for every $f \in L^2([0,1])$, the sequence $L_n(f)$ is bounded. Then by the Uniform Boundedness Theorem, the norms $\|L_n\|$ are also uniformly bounded. This contradicts our assumption.

(c) (i) **(3 marks, bookwork and homework)**
Let $x = (x_k)_{k \geq 1} \in \ell^1$. This means that $\sum_{k \geq 1} |x_k| < \infty$. Then $|x_k| \to 0$; in particular, $|x_k|^2 \leq |x_k|$ for all sufficiently large k. Hence, it follows that that $\sum_{k \geq 1} |x_k|^2 < \infty$. This shows that $\ell^1 \subset \ell^2$.
Take $x = (x_k)_{k \geq 1} \in \ell^2$. Since $\sum_{k \geq 1} |x_k|^2 < \infty$, for all $\epsilon > 0$ and $n \geq n_0(\epsilon)$ $\sum_{k \geq n+1} |x_k|^2 < \epsilon$. Take $y = (y_k)_{k \geq 1}$ such that $y_k = x_k$ for $k \leq n_0(\epsilon)$ and $y_k = 0$ otherwise. Clearly, $y_k \in \ell^1$ and $\|x - y\|_2^2 < \epsilon$. This shows that x can be approximated by elements from ℓ^1.

(ii) **(4 marks, unseen)**
Suppose that $x^{(n)} = (x_k^{(n)})_{k \geq 1} \in B_R$ and $x^{(n)} \to x$ in ℓ^2. This implies that for every k, $x_k^{(n)} \to x_k$ as $n \to \infty$. We have

$$\|x\|_1 = \lim_{N \to \infty} \sum_{k=1}^{N} |x_k| = \lim_{N \to \infty} \sum_{k=1}^{N} \lim_{n \to \infty} |x_k^{(n)}| = \lim_{N \to \infty} \lim_{n \to \infty} \sum_{k=1}^{N} |x_k^{(n)}| \leq R.$$

(iii) **(4 marks, unseen)**
Suppose that ℓ^1 has non-empty interior in ℓ^2, i.e., there exists $x \in \ell^1$ and $\epsilon > 0$ such that $x + y \in \ell^1$ for every y with $\|y\|_2 < \epsilon$. Then for every $z \in \ell^2$, $x + \frac{\epsilon}{2\|z\|_2} z \in \ell^1$. Since $x \in \ell^1$, this implies that $z \in \ell^1$. We have shown that $\ell^1 = \ell^2$, but this is not true. For instance, $x = (1/k)_{k \geq 1}$ belongs to ℓ^2, but not to ℓ^1. This contradiction implies that ℓ^1 has empty interior in ℓ^2.
Now $\ell^1 = \cup_{R=1}^{\infty} B_R$ where each B_R is closed and has empty interior, so that ℓ^1 is meager in ℓ^2.

(d) (i) **(4 marks, similar to homework)**
We write $f_n(x) = (e^{2\pi i n x} + e^{-2\pi i n x})/2$. By orthogonality for $n \neq m$,

$$\|f_n - f_m\|_2^2 = \frac{1}{4} \|e^{2\pi i n x} + e^{-2\pi i n x} - e^{2\pi i m x} - e^{-2\pi i m x}\|_2^2 = 1.$$

This implies that f_n is not a Cauchy sequence in H, so that it doesn't converge.

(ii) **(4 marks, similar to homework)**
We use that the space of trigonometric polynomials is dense in H. For every $\epsilon > 0$, there exists a trigonometric polynomial such that $\|g - p\|_2 < \epsilon$. It follows from orthogonality that for sufficiently large n, $\langle f_n, p \rangle = 0$. We obtain

$$|\langle f_n, g \rangle| \leq |\langle f_n, p \rangle| + |\langle f_n, g - p \rangle| \leq \|f_n\|_2 \|g - p\|_2 < \epsilon/\sqrt{2}.$$

This proves that $\langle f_n, g\rangle \to 0$.

End of solutions.