Functional Analysis
MATH 36202 and MATH M6202*

1 Inner Product Spaces and Normed Spaces

Inner Product Spaces

Functional analysis involves studying vector spaces where we additionally have the notion of size of an element (the norm), such spaces are known as normed spaces. Sometimes we will have an additional notion of an inner product which can be informally thought of as a way of giving an angle beteen elements. Of particular interest will be infinite dimensional spaces.

Remark. *Throughout this unit we will be taking vector spaces over either* $\mathbb{R}$ *or* $\mathbb{C}$*. We will use* $\mathbb{F}$ *to denote a field which is either* $\mathbb{R}$ *or* $\mathbb{C}$*.*

If we consider the vector space $V = \mathbb{C}^n$ then we can define the dot product for $x, y \in V$ by

$$x \cdot y = \sum_{k=1}^{n} x_k \overline{y_k}.$$

This has the following properties for all $x, y, z \in V$ and $\lambda \in \mathbb{C}$.

1. $(x + y) \cdot z = x \cdot z + y \cdot z$.
2. $\lambda(x.y) = (\lambda x).y$
3. $x \cdot y = \overline{y \cdot x}$
4. $|x| = (x \cdot x)^{1/2}$ (where $| \cdot |$ denote the Euclidean distance)
5. $|x \cdot y| \leq |x||y|$.

The notion of an inner product generalises this notion to general vector spaces.

Definition 1.1. *Let V be a vector space over $\mathbb{F}$ we say that $\langle\cdot,\cdot\rangle : V\times V \to \mathbb{F}$ is an inner product if the following properties are satisfied*

1. *For all $x, y, z \in V$ $\langle x+y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$.*

2. *For all $x, y \in V$ and $\lambda \in F$ we have that $\langle \lambda x, y\rangle = \lambda\langle x, y\rangle$.*

3. *For all $x, y \in V$ we have that $\langle x, y\rangle = \overline{\langle y, x\rangle}$.*

4. *For all $x \in V$ we have that $\langle x, x\rangle \geq 0$ with equality if and only if $x = 0$.*

When we have a vector space V over $\mathbb{F}$ with an inner product $\langle\cdot,\cdot\rangle : V\times V \to \mathbb{F}$ we'll refer to V as an inner product space over $\mathbb{F}$.

Example. We can easily check that the dot product on either $\mathbb{C}^n$ or $\mathbb{R}^n$ satisfies these properties. However if we consider the vector spaces $\mathbb{C}^{\mathbb{N}}$ or $\mathbb{R}^{\mathbb{N}}$ we cannot generalise this definition because we would not always have convergence. So for either $\mathbb{F} = \mathbb{C}$ or $\mathbb{F} = \mathbb{R}$ let

$$\ell^2 = \{\{x_n\}_{n=1}^{\infty} : x_n \in \mathbb{F} \text{ and } \sum_{n=1}^{\infty} |x_n|^2 < \infty\}.$$

We'll leave it as an exercise to check that ℓ^2 is a supspace of $\mathbb{F}^{\mathbb{N}}$. For $x, y \in \ell^2$ we let

$$\langle x, y\rangle = \sum_{n=1}^{\infty} x_n\overline{y_n}.$$

We first have to show that this is well defined. To see this we use the properties of the dot product on $\mathbb{F}^n$. We have that for any $x, y \in \ell^2$ and $k \in \mathbb{N}$ that

$$\begin{aligned} \left|\sum_{n=1}^{k} |x_n\overline{y_n}|\right| &\leq \left(\sum_{n=1}^{k} |x_n|^2\right)^{1/2} \left(\sum_{n=1}^{k} |y_n|^2\right)^{1/2} \\ &\leq \left(\sum_{n=1}^{\infty} |x_n|^2\right)^{1/2} \left(\sum_{n=1}^{\infty} |y_n|^2\right)^{1/2}. \end{aligned}$$

Thus this is bounded independently of k and so is absolutely convergent. Thus the definition makes sense. It is now routine to check that this is an inner product.

We can now state some simple properties of the inner product

Theorem 1.2. *Let V be an inner product space over $\mathbb{F}$. We have that*

1. *For $x, y, z \in V$ that $\langle x, y+z\rangle = \langle x, y\rangle + \langle x, z\rangle$.*

2. *For $x, y \in V$ and $\lambda \in F$ that $\langle x, \lambda y\rangle = \overline{\lambda}\langle x, y\rangle$*

3. For any $x \in V$ we have that $\langle x, 0\rangle = \langle 0, x\rangle = 0$.

4. If $x, z \in V$ satisfies that $\langle x, y\rangle = \langle z, y\rangle$ for all $y \in V$ then $x = z$.

Proof. Let $x, y, z \in V$ and $\lambda \in \mathbb{F}$. For part 1 we have

$$\langle x, y + z\rangle = \overline{\langle y + z, x\rangle} = \overline{\langle y, x\rangle + \langle z, x\rangle} = \langle x, z\rangle + \langle y, z\rangle.$$

For part 2

$$\langle x, \lambda y\rangle = \overline{\langle \lambda y, x\rangle} = \overline{\lambda\langle y, x\rangle} = \overline{\lambda}\overline{\langle y, x\rangle} = \overline{\lambda}\langle x, y\rangle.$$

For part 3 we have that

$$\langle x, 0\rangle = \langle x, 0 \cdot x\rangle = 0\langle x, x\rangle = 0 = 0\langle x, x\rangle = \langle 0, x\rangle.$$

Finally $\langle x, y\rangle = \langle z, y\rangle$ for all $y \in V$ implies that $\langle x - z, y\rangle = 0$ for all $y \in V$. Thus $\langle x - z, x - z\rangle = 0$ and so $x - z = 0$ which means $x = z$. $\square$

We can use the notion of an inner product to give a notion of size and distance on V.

Definition 1.3. *For $v \in V$ we define*

$$\|v\| = (\langle v, v\rangle)^{1/2}.$$

Theorem 1.4 (Cauchy-Schwarz inequality)**.** *Let V be an inner product space over $\mathbb{F}$. We have that for all $x, y \in V$*

$$|\langle x, y\rangle| \leq \|x\|\|y\|.$$

Moreover, when x and y are linearly independent, the inequality is strict.

Proof. We first suppose there exists $\lambda \in \mathbb{F}$ such that $x = \lambda y$. We then have that

$$|\langle \lambda y, y\rangle| = |\lambda|\langle y, y\rangle = (\lambda\overline{\lambda}\langle y, y\rangle\langle y, y\rangle)^{1/2} = (\langle \lambda y, \lambda y\rangle\langle y, y\rangle)^{1/2} = \|y\|\|\lambda y\|.$$

Now we suppose that x and y are linearly independent. So for all $\lambda \in \mathbb{F}$ we have that $\langle y + \lambda x, y + \lambda x\rangle > 0$. We can write

$$\begin{array}{rcl} 0 & < & \langle y + \lambda x, y + \lambda x\rangle \\ & = & \|y\|^2 + \langle \lambda x, y\rangle + \langle y, \lambda x\rangle + |\lambda|^2\|x\|^2 \\ & = & \|y\|^2 + 2\mathrm{Re}(\langle \lambda x, y\rangle) + |\lambda|^2\|x\|^2 \\ & = & \|y\|^2 + 2\mathrm{Re}(\lambda\langle x, y\rangle) + |\lambda|^2\|x\|^2 \end{array}$$

We now let $\lambda = tu$ where $t \in \mathbb{R}$ and $u = \frac{|\langle x,y\rangle|}{\langle x,y\rangle}$, note that $|u| = 1$. Substituting this in gives that

$$0 > \|y\|^2 + 2t|\langle x, y\rangle| + t^2\|x\|^2.$$

Thus the quadratic $\|y\|^2 + 2t|\langle x, y\rangle| + t^2\|x\|^2$ has no roots and so must have discriminant less than 0. Thus

$$4|\langle x, y\rangle|^2 < 4\|x\|^2\|y\|^2$$

and the result follows. $\square$

We now can deduce that $\|\cdot\|$ has the properties we would wish a notion of size to have.

Theorem 1.5. *Let V be an inner product space over $\mathbb{F}$. For $v \in V$ let $\|v\| = (\langle v, v\rangle)^{1/2}$. We have that*

1. *For all $v \in V$ $\|v\| > 0$ unless $v = 0$.*
2. *For all $v \in V$ and $\lambda \in \mathbb{F}$ we have $\|\lambda v\| = |\lambda|\|v\|$.*
3. *For all $x, y, z \in V$ we have that*
$$\|x + y\| \leq \|x\| + \|y\|$$
4. *If we define $d(x, y) = \|x - y\|$ then this gives a metric on V.*

Proof. The first and fourth parts are obvious. For the second part write

$$\|\lambda v\| = (\langle \lambda v, \lambda v\rangle)^{1/2} = (\lambda\overline{\lambda}\langle v, v\rangle)^{1/2} = \lambda\|v\|.$$

Finally for the third part we use Cauchy-Schwarz. We fix $x, y \in V$ and write

$$\begin{aligned} \|x + y\|^2 &= \langle x + y, x + y\rangle \\ &= \langle x, x\rangle + \langle y, y\rangle + 2\mathrm{Re}(\langle x, y\rangle) \\ &\leq \langle x, x\rangle + \langle y, y\rangle + 2|\langle x, y\rangle| \\ &\leq \|x\|^2 + \|y\|^2 + 2\|x\|\|y\| = (\|x\| + \|y\|)^2. \end{aligned}$$

$\square$

Example. Let V be the vector space of all continuous functions $f : [0, 1] \to \mathbb{F}$. We can define

$$\langle f, g\rangle = \int_0^1 f\overline{g}\mathrm{d}x.$$

This gives an inner product space (see exercises).

We have seen that an inner product gives a way of definining a size and a distance between elements. This means inner product spaces give examples of what are called normed spaces, however not all normed spaces come from inner products.

Normed spaces

Definition 1.6. *Let V be a vector space over $\mathbb{F}$ we say that $\|\cdot\| : V \to \mathbb{R}$ is a norm if*

1. $\|v\| \geq 0$ *for all $v \in V$ with equality if and only if $v = 0$.*
2. *For all $v \in V$ and $\lambda \in \mathbb{F}$ we have that $\|\lambda v\| = |\lambda| \|v\|$.*
3. *For all $x, y \in V$ we have that*

$$\|x + y\| \leq \|x\| + \|y\|.$$

We then call V a normed space with respect to $\|\cdot\|$.

Example. Consider $C([0,1])$ the space of real valued continuous functions (we could also take complex valued functions). We define

$$\|f\| = \sup_{x \in [0,1]} \{|f(x)|\}.$$

We can easily see this gives a norm. Note that if we use the inner product

$$\langle f, g \rangle = \int_0^1 fg \mathrm{d}x$$

then we get a different norm

$$\|f\|_2 = (\int_0^1 |f(x)|^2)^{1/2}.$$

Definition 1.7 (ℓ^p space)**.** *We now consider certain spaces of sequences on which there will be a natural definition of a norm. Let $1 \leq p < \infty$ and define*

$$\ell^p = \{\{x_n\}_{n \in \mathbb{N}} : x_n \in \mathbb{F} \text{ and } \sum_{n=1}^{\infty} |x_n|^p < \infty\} \subset \ell^p.$$

For $x \in \ell^p$ we define

$$\|x\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{1/p}.$$

Note that for $p = 2$ this agrees with the norm induced by the inner product. We can also define

$$\ell^\infty = \{\{x_n\}_{n \in \mathbb{N}} : x_n \in F \text{ and } \sup_{n \in \mathbb{N}} |x_n| < \infty\}$$

and

$$|x\|_\infty = \sup_{n \in \mathbb{N}} \{\|x_n|\}.$$

Theorem 1.8. *For all $1 \leq p \leq \infty$ we have that ℓ^p is a normed space.*

Proof. It is an exercise to show that ℓ^p is a subspace of $\mathbb{F}^{\mathbb{N}}$. It is clear that for all $1 \leq p \leq \infty$ that for nonzero $x \in \ell^p$, $\|x\| > 0$ and that for all $\lambda \in \mathbb{F}$ we have $\|\lambda x\| = |\lambda|\|x\|$. So we need to show the triangle inequality. For $p = 1$ and $p = \infty$ it is easy to show.

So now let $1 < p < \infty$ and $q = \frac{p}{p-1}$. We have Young's inequality (see exercise sheet) that for all $A, B \geq 0$ that

$$AB \leq \frac{A^p}{p} + \frac{B^q}{q}.$$

Hölder's inequality

We now show that for $x, \in \ell^p$ and $y \in \ell^q$ that

$$\sum_{k=1}^{\infty} |x_k y_k| \leq \|x\|_p \|y\|_q.$$

To do this fix $x, y \in \ell^p$ and $k \in \mathbb{N}$. We use $A = \frac{|x_k|}{\|x\|_p}$ and $B = \frac{|y_k|}{\|y\|}$ in Young's inequality to get

$$\frac{|x_k||y_k|}{\|x\|_p \|y\|_q} \leq \frac{1}{p} \frac{|x_k|^p}{\|x\|_p^p} + \frac{1}{q} \frac{|y_k|^q}{\|y\|_q^q}.$$

We can now sum over all $k \in \mathbb{N}$ to get

$$\frac{1}{\|x\|_p \|y\|_q} \sum_{k=1}^{\infty} |x_k y_k| \leq \frac{1}{p} + \frac{1}{q}$$

and finally

$$\sum_{k=1}^{\infty} |x_k y_k| \leq \|x\|_p \|y\|_q.$$

Minkowski's inequality

We can now prove the triangle inequality for ℓ^p also known as Minkowski's inequality. That is for all $x, y \in \ell^p$

$$\|x + y\|_p \leq \|x\|_p + \|y\|_p.$$

To do this, we first prove this inequality in $\mathbb{F}^N$. Using the Hölder inequality,

$$\begin{aligned}
\sum_{k=1}^{N} |x_k + y_k|^p &= \sum_{k=1}^{N} |x_k + y_k||x_k + y_k|^{p-1} \\
&\leq \sum_{k=1}^{N} |x_k||x_k + y_k|^{p-1} + \sum_{k=1}^{N} |y_k||x_k + y_k|^{p-1} \\
&\leq \left(\sum_{k=1}^{N} |x_k + y_k|^p \right)^{1/q} (\|x\|_p + \|y\|_p).
\end{aligned}$$

We can now rearrange this and use that $p - p/q = 1$ to get

$$\left(\sum_{k=1}^{N} |x_k + y_k|^p\right)^{1/p} \leq \left(\sum_{k=1}^{N} |x_k|^p\right)^{1/p} + \left(\sum_{k=1}^{N} |y_k|^p\right)^{1/p}.$$

Now we take $N \to \infty$. $\square$

Remark. *ℓ^p space will be one of the main examples of a normed space we'll use. For $p = 2$ we have already seen that it is also an inner product space.*

Remark. *On a normed space V, we have a metric given by $d(x, y) = \|x-y\|$. When we refer to concepts from metric spaces it is with respect to this metric (e.g. continuity, convergence (you'll see other notions of this later on), open and closed sets).*

Example. Let V be a normed space, $x \in V$ and define $f_x : V \to \mathbb{R}$ by

$$f_x(y) = \|x + y\|.$$

f_x is a continuous map on V since if we fix $y \in V$, $\epsilon > 0$ and choose $\delta = \epsilon$. We then have that if $\|z - y\| \leq \delta$ then

$$|f_x(y) - f_x(z)| = |\|y + x\| - \|x + z\||.$$

Using the triangle inequality we have that for any $a, b \in V$

$$\|a\| - \|b\| \leq \|a - b\|,$$

and

$$-(\|a\| - \|b\|) = \|b\| - \|a\| \leq \|b - a\|.$$

Hence,

$$|\|a\| - \|b\|| \leq \|a - b\|.$$

Thus combining this inequality with the triangle inequality

$$|f_x(y) - f_x(z)| = |\|y + x\| - \|x + z\|| \leq \|y - z\| \leq \epsilon.$$

Subspaces and closed subspaces

If we take $\mathbb{R}^n$ or $\mathbb{C}^n$ with the Euclidean norm it is straightforward to see that any subspace is closed. However this is not the case for infinite dimensional normed spaces.

Example. Take $V = \ell^2$ and let $A \subset \ell^2$ consider of all sequence where only finitely many terms are nonzero. We can easily see that A is a subspace of ℓ^2. However let $x_n \in A$ be written $x_n^{(1)}, x_n^{(2)}, \ldots$ with $x_n^{(k)} = \frac{1}{k}$ if $k \leq n$ and

$x_n^{(k)} = 0$ otherwise. Let $y \in \ell^2$ satisfy $y_k = \frac{1}{k}$ for all $k \in \mathbb{N}$. We can see that $y \notin A$ but

$$\|x_n - y\|_2 = \sum_{k=n+1}^{\infty} \frac{1}{k^2} \to 0 \text{ as } n \to \infty.$$

So A is not a closed subspace of ℓ^2.

Theorem 1.9. *Let V be a normed space and A be a subspace of V. The closure of A is a subspace of V.*

Proof. Let A be a subspace of V and let $\overline{A}$ denote the closure of A. We let $x, y \in \overline{A}$ and $\lambda \in F \backslash \{0\}$. Thus there exist sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ in A such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$. Let $\epsilon > 0$ and choose N such that for all $n \geq N$ we have that $\max\{\|x_n - x\|, \|y_n - y\|\} \leq \epsilon/2$. Thus for $n \geq N$ we have that

$$\|x_n + y_n - x + y\| \leq \|x_n - x\| + \|y_n - y\| \leq 2.$$

Thus $\lim_{n\to\infty} x_n + y_n = x + y$ and so $x + y \in \overline{A}$. Finally if we choose N such that for $n \geq N$ we have $\|x_n - x\| \leq \epsilon/|\lambda|$ then for $n \geq N$,

$$\|\lambda(x_n - x)\| = |\lambda|\|x_n - y\| \leq \epsilon.$$

Thus $\lambda x \in \overline{A}$ and we can conclude that $\overline{A}$ is a subspace of V. $\square$

We finish this section by defining the closed (linear) span of any subset of a normed space.

Definition 1.10. Let V be a normed space and $A \subset V$. We define the closed (linear) span of A to be the closure of the (linear) span of A. The closed span of A will be a closed subspace of V.