10 The Open Mapping Theorem and the Closed Graph Theorem

10.1 The Open Mapping Theorem

We recall that a map $f : X \to Y$ between metric spaces in continuous if and only if the preimages $f^{-1}(U)$ of all open sets in Y are open in X.

Definition 10.1 (open mapping)**.** Let X, Y be metric spaces. A map $f : X \to Y$ is called an *open mapping* if for all open $U \subset X$, the sets $f(U)$ are open in Y. In other words, "f takes open sets to open sets."

Example 10.2.

1. The projection $\pi : \mathbb{R}^n \to \mathbb{R}^k$ defined by

$$(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_k),$$

where $k < n$, is open. Let $U \subset \mathbb{R}^n$ be an open set, and consider its image $\pi(U) \subset \mathbb{R}^k$. Let $y \in T(U)$. Then $y = T(x)$ for some $x = (x_1, \ldots, x_n) \in U$. There is an open set of the form $B_\epsilon(x) = (x_1 - \epsilon, x_1 + \epsilon) \times \cdots \times (x_n - \epsilon, x_n + \epsilon)$. Then $\pi(B_\epsilon(x)) = (x_1 - \epsilon, x_1 + \epsilon) \times \cdots \times (x_k - \epsilon, x_k + \epsilon)$ is an open set contained in $\pi(U)$ and containing the given point y, so $\pi(U)$ is open and the projection π is an open mapping.

2. Inclusion map $\mathbb{R}^k \hookrightarrow \mathbb{R}^n$, defined by

$$(x_1, \ldots, x_k) \mapsto (x_1, \ldots, x_k, 0, \ldots, 0),$$

where $k < n$, is not open. It is clear that any open ball around a point in the image of T will contain points that are not in the image of T, so this map cannot be open.

3. Sine map $\sin : \mathbb{R} \to \mathbb{R}$ is not open. The image of a long enough interval is the closed interval $[-1, 1]$, which is not open, so sin is not an open mapping.

Theorem 10.3 (Open Mapping Theorem)**.** *Let X, Y be Banach spaces and $T : X \to Y$ a surjective bounded linear map. Then T is open.*

Proof. Let

$$B_r^X = \{x \in X : \|x\| < r\} \quad \text{and} \quad B_r^Y = \{y \in Y : \|y\| < r\}.$$

First, we show that there exists $r > 0$ such that

$$B_r^Y \subset T(B_1^X). \tag{1}$$

For this, we use the Baire Category Theorem. We have

$$X = \bigcup_{n=1}^{\infty} B_n^X,$$

and since T is surjective,

$$Y = \bigcup_{n=1}^{\infty} T(B_n^X).$$

We have also assumed that Y is Banach, hence complete. By the Baire Category Theorem, there must be some $n_0 \in \mathbb{N}$ such that $\overline{T(B_{n_0}^X)}$ has non-empty interior. This implies that $\overline{T(B_1^X)} = n_0^{-1}\,\overline{T(B_{n_0}^X)}$ also has non-empty interior. Hence, there exists $y_0 \in \overline{T(B_1^X)}$ and $\epsilon > 0$ such that

$$y_0 + B_\epsilon^Y \subset \overline{T(B_1^X)}. \tag{2}$$

We claim that

$$\overline{T(B_1^X)} - y_0 \subset \overline{T(B_2^X)}. \tag{3}$$

Take $y \in \overline{T(B_1^X)}$ and $x_n \in X$ such that $Tx_n \to x$ and $\|x_n\| < 1$. We also have a sequence $z_n \in X$ such that $Tz_n \to y_0$ and $\|z_n\| < 1$. Then $y - y_0 = \lim T(x_n - z_n)$ and $\|x_n - z_n\| < 1$. This implies (3). Combining (2) and (3), we deduce that

$$B_\epsilon^Y \subset \overline{T(B_2^X)}. \tag{4}$$

Furthermore, because T is linear, we also have

$$B_{\epsilon/2^n}^Y \subset \overline{T(B_{1/2^{n-1}}^X)} \quad \text{for all } n \geq 1. \tag{5}$$

Take arbitrary $y \in B_{\epsilon/8}^Y$. Then $y \in \overline{T(B_{1/4}^X)}$. So that we can find $x_1 \in B_{1/4}^X$ such that $\|y - Tx_1\| < \frac{\epsilon}{8}$, that is, $y - Tx_1 \in B_{\epsilon/8}^Y$. Applying (5) again, we deduce that there exists $x_2 \in B_{1/8}^X$ such that $\|y - Tx_1 - Tx_2\| < \frac{\epsilon}{16}$, that is, $y - Tx_1 \in B_{\epsilon/16}^Y$. Proceeding inductively, we produce elements $x_k \in B_{1/2^{k+1}}^X$ such that

$$\left\| y - \sum_{k=1}^{n} T(x_k) \right\| < \frac{\epsilon}{2^{n+2}}. \tag{6}$$

Now let $z_n = x_1 + \cdots + x_n$. For $m < n$,

$$\|z_n - z_m\| = \|x_{m+1} + \cdots + x_n\| \leq \sum_{k=m+1}^{n} \|x_k\| < \sum_{k=m+1}^{n} \frac{1}{2^{k+1}} < \frac{1}{2^{m+1}}.$$

Hence, $(z_n)_{n\geq 1}$ is a Cauchy sequence, and $z_n \to z$ for some $z \in X$. We note that

$$\|z\| = \lim_{n\to\infty} \|z_n\| \leq \lim_{n\to\infty} \sum_{k=1}^{n} \|x_k\| \leq \sum_{k=1}^{\infty} \frac{1}{2^{k+1}} = \frac{1}{2},$$

so that $x \in B_1^X$. Now continuity of T and equation (6) imply that $T(z) = y$. Thus, we have proved that $B_{\epsilon/8}^Y \subset T(B_1^X)$ verifying (1).

Now we complete the proof of the theorem using (1). Let $U \subset X$ be open. We want to show that $T(U)$ is also open. So take $y \in T(U)$. Then $y = Tx$ with $x \in U$. Since U is open, there exists $\epsilon > 0$ such that $B_\epsilon^X + x \subset U$ By (1) and linearity of T, we obtain $B_{r\epsilon}^Y \subset T(B_\epsilon^X)$, and

$$B_{r\epsilon}^Y + y \subset T(B_\epsilon^X + x) \subset T(U).$$

This shows that $T(U)$ is open, as required. □

We deduce the following corollary

Corollary 10.4. *Let $T : X \to Y$ be a continuous (bounded) linear bijection between Banach spaces X and Y. Then T^{-1} is continuous (bounded).*

This implies the following surprising consequence:

Corollary 10.5. *Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be norms on a vector space X such that $(X, \|\cdot\|_1)$ and $(X, \|\cdot\|_2)$ are complete. Then if*

$$\|\cdot\|_2 \leq c\,\|\cdot\|_1 \quad \text{for some } c > 0,$$

then also

$$\|\cdot\|_1 \leq c'\,\|\cdot\|_2 \quad \text{for some } c' > 0,$$

Indeed, then the identity map $(X, \|\cdot\|_1) \to (X, \|\cdot\|_2)$ is bounded, and it follows from Corollary 10.4 that its inverse is also bounded.

10.2 Closed linear operators

So far we have mostly discussed bounded linear operator. Many of the important classes of operators (for instance, differential operators) are unbounded. We introduce the so-called closed operators.

Definition 10.6 (closed linear operator)**.** Let X, Y be normed spaces, $\mathscr{D}$ a subspace of X, and $T : \mathscr{D} \to Y$ a linear operator. The operator T is *closed* if its graph

$$\Gamma(T) = \{(x, Tx) \in X \times Y : x \in \mathscr{D}\}$$

is closed in the normed space $X \times Y$.

This means that for every sequence $x_n \in \mathscr{D}$ such that $x_n \to x$ and $Tx_n \to y$, we have $x \in \mathscr{D}$ and $Tx = y$.

Example 10.7. Let $X = C[0,1]$ be the space of continuous functions equipped with max-norm. We consider the derivative operator $D : \mathscr{D} \to X$ be defined by $D(x)(t) = x'(t)$ for $x \in \mathscr{D} = C^1[0,1]$ in the space of continuously differentiable functions.

The operator D is an unbounded operator. Indeed, let $x_n(t) = t^n$, for $n \geq 1$. Then $\|x\| = 1$. Also, $D(x_n) = x_n'(t) = nt^{n-1}$, and we can see that $\|D(x_n)\| = n$. Therefore, $\|D\| \geq n$ for any $n \geq 1$, so D is an unbounded operator.

We show that D is a closed operator. Let $x_n \in \mathscr{D}$ be a sequence converging to $x \in C[0,1]$, and such that $x_n' \to y \in C[0,1]$. Since the convergence is uniform, we obtain

$$\begin{aligned}\int_0^t y(\tau)\,d\tau &= \int_0^t \lim_{n\to\infty} x_n'(\tau)\,d\tau = \lim_{n\to\infty}\int_0^t x_n'(\tau)\,d\tau = \lim_{n\to\infty}(x_n(t) - x_n(0)) \\ &= x(t) - x(0).\end{aligned}$$

This shows that $x(t) = x(0) + \int_0^t y(\tau)\,d\tau$. Therefore, $x \in C^1[0,1]$, and $x' = y$. We conclude that D is a closed linear operator.

Another source of examples of closed operators is Hilbert-adjoint operators:

Definition (Hilbert adjoints)**.** Let X be a Hilbert space. Let $T : \mathscr{D} \to X$ be a linear operator defined on a dense subspace $\mathscr{D}$ of X is dense. Then the *Hilbert-adjoint operator* of T is the operator $T^* : \mathscr{D}^* \to X$ with

$$\mathscr{D}^* = \{y \in Y : \exists y^* \in X \text{ such that } \langle Tx, y\rangle = \langle x, y^*\rangle \text{ for all } x \in \mathscr{D}\}$$

defined by $T^*(y) = y^*$.

We note that the Hilbert-adjoint operator satisfies

$$\langle Tx, y\rangle = \langle x, T^*y\rangle \quad \text{for all } x \in \mathscr{D} \text{ and } y \in \mathscr{D}^*.$$

Theorem 10.8. *Hilbert-adjoint operators are closed.*

Proof. Suppose $y_n \in \mathscr{D}^*$ converges to $y \in X$, and T^*y_n converges to $z \in X$. Since $y_n \in \mathscr{D}^*$, we know that for any $x \in \mathscr{D}$, $\langle Tx, y_n\rangle = \langle x, T^*y_n\rangle$. Passing to the limit, we obtain $\langle Tx, y\rangle = \langle x, z\rangle$. This means that $y \in \mathscr{D}^*$ and $T^*y = z$. Hence, T^* is closed. □

10.3 The Closed Graph Theorem

The following theorem treats the question of when a bounded operator is closed.

Theorem 10.9. *Let X and Y be normed spaces. Let $T : \mathscr{D} \to Y$ be a bounded linear operator defined on a subspace $\mathscr{D}$ of X.*

1. *If $\mathscr{D}$ is closed in X, then T is closed.*

2. *If T is closed and Y is complete, then $\mathscr{D}$ is closed in X.*

Proof. Suppose $x_n \in \mathscr{D}$ converges to $x \in X$ and $Tx_n \to y \in Y$. Since we have assumed that $\mathscr{D}$ is closed, we must have $x \in \mathscr{D}$. Since T is bounded, it is continuous, and therefore $y = Tx$. This proves that T is closed.

Suppose that T is closed and Y is complete. Let $x \in \overline{\mathscr{D}}$ and $x_n \in \mathscr{D}$ such $x_n \to x$. Since

$$\|Tx_m - Tx_n\| \leq \|T\| \, \|x_m - x_n\|,$$

it follows that Tx_n is a Cauchy sequence. Therefore, $Tx_n \to y$ for some $y \in Y$. Since T is closed, we conclude that $x \in \mathscr{D}$. Hence, $\mathscr{D}$ is closed. □

The following theorem gives a condition for when a closed linear operator is bounded.

Theorem 10.10 (Closed Graph Theorem)**.** *Let X, Y be Banach spaces and $T : \mathscr{D} \to Y$ a closed linear operator defined on a closed subspace $\mathscr{D} \subset X$. Then T is bounded.*

To appreciate the Closed Graph Theorem, we observe that to check that a map $T : X \to Y$ is continuous, one normally need to show that

$$x_n \to x \quad \Rightarrow \quad Tx_n \to y \text{ and } Tx = y.$$

In the setting of the Closed Graph Theorem, continuity follows once we show that

$$x_n \to x \text{ and } Tx_n \to y \quad \Rightarrow \quad Tx = y.$$

We easily obtain the following corollary.

Corollary 10.11. *Let $T : H \to H$ be an linear operator on a Hilbert space H such that $\langle Tx, y \rangle = \langle x, Ty \rangle$ for all $x, y \in H$. Then T is bounded.*

Indeed, the operator T is closed by Theorem 10.8.

Proof of Theorem 10.10. We first note that the space $X \times Y$ equipped with $\|(x,y)\| = \|x\| + \|y\|$ is complete (exercise). Since the graph $\Gamma(T)$ is closed in $X \times Y$, it is a Banach space. Since the domain $\mathscr{D}$ is closed, it is also a Banach space.

We consider the map

$$P : \Gamma(T) \to \mathscr{D} : (x, Tx) \mapsto x.$$

Clearly, P is linear and a bijection. We have $P^{-1}x = (x, Tx)$. Since

$$\|P(x, Tx)\| = \|x\| \leq \|x\| + \|Tx\| = \|(x, Tx)\|,$$

it is also bounded. By the Open Mapping Theorem, the inverse P^{-1} is bounded, meaning that there is an $C > 0$ such that for any $x \in \mathscr{D}$,

$$\|P^{-1}x\| = \|x\| + \|Tx\| \leq C\,\|x\|.$$

Hence, T is bounded. □