2 Banach spaces and Hilbert spaces

Trying to do analysis in the rational numbers is difficult for example consider the set $\{x \in \mathbb{Q} : x^2 \leq 2\}$. This set is non-empty and bounded above but does not have a least upper bound in the rationals. In the real numbers things are much better since we have completeness (one of the axioms for the real numbers) which says that every set of reals bounded above has a least upper bound. An alternative way to state this axiom is that every Cauchy sequence in the real numbers is convergent to a real number. Similarly with normed spaces it will be easier to work with spaces where every Cauchy sequence is convergent. Such spaces are called Banach spaces and if the norm comes from an inner product then they are called Hilbert spaces.

Definition 2.1. *Let V be a normed space with norm $\|\cdot\|$ and $\{x_n\}_{n\in\mathbb{N}}$ a sequence in V. We say that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence if for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n, m \geq N$ we have that*

$$\|x_n - x_m\| \leq \epsilon.$$

Definition 2.2. *Let V be a normed space with norm $\|\cdot\|$. We say that V is a Banach space if every Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ in V is convergent (i.e. there exists $x \in V$ such that $\lim_{n\to\infty}\|x_n - x\| = 0$). In other words the metric defined by the norm is complete.*

Definition 2.3. *Let V be an inner product space. We say that V is a Hilbert space if V is a Banach space with the norm defined by the inner product (i.e. $\|v\| = \sqrt{\langle v, v\rangle}$).*

In this rest of the section we will look at some examples of normed spaces which are Banach spaces and some which are not. We'll then go on to look at the geometry of Hilbert spaces in more detail. To show that a normed space V is a Banach space we simply need to show that every Cauchy sequence x_n is convergent. To do this there is a standard stragedy.

step 1 Find a candidate y for the limit of x_n.

step 2 Show that $y \in V$.

step 3 Show that $\lim_{n\to\infty} x_n = y$.

Example. Let $V = C([0, 1])$ be the space of all complex valued continuous functions with the norm $\|f\|_\infty = \sup_{x\in[0,1]\{|f(x)|}$. We show that V is a Banach space using the above stragedy. Let f_n be a Cauchy sequence in V.

step 1. We need to find a candidate f for the limit of f_n. Fix $x \in [0, 1]$ and note that for all $n, m \in \mathbb{N}$ we have that

$$|f_n(x) - f_m(x)| \leq \|f_n - f_m\|_\infty.$$

Thus $f_n(x)$ is a Cauchy sequence in $\mathbb{C}$ and since $\mathbb{C}$ is complete we know that $\lim_{n\to\infty} f_n(x)$ exists. So we let $f(x) = \lim_{n\to\infty} f_n(x)$.

step 2. We need to show that f is continuous. Fix $x \in [0,1]$ and let $\epsilon > 0$ we can choose N such that for all $n, m \geq N$ we have that $\|f_n - f_m\|_\infty \leq \epsilon/3$. We fix $\delta > 0$ such that if $|x-y| \leq \delta$ then $|f_N(x) - f_N(y)| \leq \epsilon/3$. For any $n, m \geq N$ and y such that $|x-y| \leq \delta$ we have that

$$\begin{aligned} & |f(x) - f(y)| \\ = \ & |f(x) - f_n(x) + f_n(x) - f_N(x) + f_N(x) - f_N(y) + f_N(y) - f_m(y) + f_m(y) - f(y)| \\ \leq \ & |f(x) - f_n(x)| + \|f_n - f_N\|_\infty + |f_N(x) - f_N(y)| + |f_N(y) - f_m(y)| + |f_m(y) - f(y)| \\ \leq \ & 3\epsilon + |f(x) - f_n(x)| + |f_m(y) - f(y)|. \end{aligned}$$

If we let $n, m \to \infty$ and use that $0 = \lim_{n\to\infty} |f(x) - f_n(x)| = \lim_{m\to\infty} |f_m(y) - f(y)|$ then we have that

$$|f(x) - f(y)| \leq \epsilon$$

and we can conclude that f is continuous.

step 3 Finally we need to show that $\lim_{n\to\infty}\|f_n - f\|_\infty = 0$. Again let $\epsilon > 0$ and choose N such that for $n, m \geq N$ we have that $\|f_n - f_m\| \leq \epsilon$. For $n, m \geq N$ and any $x \in [0,1]$ we can write

$$|f_n(x) - f(x)| \leq |f_n(x) - f_m(x)| + |f_m(x) - f(x)| \leq \epsilon + |f_m(x) - f(x)|.$$

So if we let $m \to \infty$ we get that $|f_n(x) - f(x)| \leq \epsilon$. Since this holds for all $x \in [0,1]$ we have that $\|f_n - f\| \leq \epsilon$ and we can conclude tha tin V, $\lim_{n\to\infty} f_n = f$. Thus V is a Banach space.

Theorem 2.4. *For all $1 \leq p \leq \infty$ we have that ℓ^p with respect to the norm $\|\cdot\|$ is a Banach space.*

Proof. The case $p = \infty$ is left a san exercise so we'll fix $1 \leq p < \infty$ and follow the same three steps. We let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in ℓ^p and write

$$x_n = (x_n^{(1)}, x_n^{(1)}, \ldots).$$

step 1. Fix $k \in \mathbb{N}$ and consider the sequence $(x_n^{(k)})_{n\in\mathbb{N}}$ in F. For any $n, m \in \mathbb{N}$ we have that

$$|x_n^{(k)} - x_m^{(k)}| = (|x_n^{(k)} - x_m^{(k)}|^p)^{1/p} \leq \left(\sum_{k=1}^{\infty} |x_n^{(k)} - x_m^{(k)}|^p \right)^{1/p} = \|x_n - x_m\|_p.$$

Thus $x_n^{(k)}$ is a Cauchy sequence in $\mathbb{F}$ and so has a limit $y^{(k)}$. So let $y = (y^{(1)}, y^{(2)}, \ldots)$.

step 2. We now need to show that $y \in \ell^p$. So we need to show that $\sum_{k=1}^{\infty} |y^{(k)}|^p < \infty$. We let $N \in \mathbb{N}$ satisfy that for any $n, m \geq N$ we have $\|x_n - x_m\| \leq \epsilon$. Thus for $j, n \in \mathbb{N}$ with $n \geq N$ we have that by Minkowski's inequality

$$\begin{aligned} \left(\sum_{k=1}^{j} |y^{(k)}|^p \right)^{1/p} &= \left(\sum_{k=1}^{j} |y^{(k)} - x_n^{(k)} + x_n^{(k)} - x_N^{(k)} + x_N^{(k)}|^p \right)^{1/p} \\ &\leq \left(\sum_{k=1}^{j} |y^{(k)} - x_n^{(k)}|^p \right)^{1/p} + \|x_n - x_N\|_p + \|x_N\|_p. \end{aligned}$$

We have that for any $n \geq N$, $\|x_n - x_N\| \leq 1$ and

$$\lim_{n \to \infty} \left(\sum_{k=1}^{j} |y^{(k)} - x_n^{(k)}|^p \right)^{1/p} = 0.$$

So we can conclude that for any $j \in \mathbb{N}$,

$$\left(\sum_{k=1}^{j} |y^{(k)}|^{(p)} \right)^{1/p} \leq 1 + \|x_N\|_p$$

and thus

$$\sum_{k=1}^{\infty} |y^{(k)}|^p < \infty.$$

step 3 Finally we have to show that $\lim_{n \to \infty} \|x_n - y\|_p = 0$ and we proceed similarly to step 2. Let $\epsilon > 0$ and choose N such that for all $n, m \geq N$ we have that $\|x_n - x_m\|_p \leq \epsilon$. For $n, m \geq N$ and $j \in \mathbb{N}$ we have that using Minkowski's inequality

$$\left(\sum_{k=1}^{j} |y^{(k)} - x_n^{(k)}|^p \right)^{1/p} \leq \left(\sum_{k=1}^{j} |y^{(k)} - x_m^{(k)}|^p \right)^{1/p} + \|x_m - x_n\|_p.$$

For any $n, m \geq N$ we have that $\|x_n - x_m\| \leq \epsilon$ and

$$\lim_{m \to \infty} \left(\sum_{k=1}^{j} |y^{(k)} - x_m^{(k)}|^p \right)^{1/p} = 0.$$

Thus

$$\left(\sum_{k=1}^{j} |y^k - x_n^{(k)}|^p \right)^{1/p} \leq \epsilon$$

and since this bound is independent of j we have that $\|y - x_n\|_p \leq \epsilon$. Thus we have shown that $\lim_{n \to \infty} \|x_n - y\|_p = 0$.

□

We now turn to an example which is not a Banach space.

Example. Let $C([0,1])$ be the space of continuous functions $f : [0,1] \to \mathbb{C}$. We define

$$\|f\|_2 = \left(\int_0^1 |f(x)|^2 \, dx \right)^{1/2}.$$

This is a normed space where the norm comes from an inner product but it not a Banach space. To see this we define for $n \geq 2$

$$f_n(x) = \begin{cases} 0 & \text{if} \quad 0 \leq x \leq \frac{1}{2} - \frac{1}{n} \\ n\left(x - \frac{1}{2} + \frac{1}{n}\right) & \text{if} \quad \frac{1}{2} - \frac{1}{n} < x \leq \frac{1}{2} \\ 1 \text{ if } \frac{1}{2} < x \leq 1. \end{cases}$$

Let $\epsilon > 0$ choose N such that $1/\sqrt{N} \leq \epsilon$. We then have that for $n, m \geq N$

$$\int_0^1 |f_n(x) - f_m(x)|^2 \mathrm{d}x \leq \int_{1/2-1/N}^{1/2} 1 \mathrm{d}x = 1/N.$$

Thus $\|f_n - f_m\|_2 \leq 1/\sqrt{N} \leq \epsilon$. So f_n is a Cauchy sequence. Now suppose there exists $f \in C([0,1])$ such that $\lim_{n\to\infty} \|f_n - f\|_2 = 0$. In this case we have that

$$\lim_{n\to\infty} \int_0^{1/2} |f_n(x) - f(x)|^2 \mathrm{d}x = \lim_{n\to\infty} \int_{1/2}^{1} |f_n(x) - f(x)|^2 \mathrm{d}x = 0.$$

Since $f_n(x)$ is constantly equal to 1 on $[1/2, 1]$ this means that $f(x) = 1$ for all x on $[1/2, 1]$. Now since f is continuous this means there exists $\delta > 0$ such that $f(x) > 1/2$ for all $x \in [1/2 - \delta, 1/2]$. We now choose $N \in \mathbb{N}$ such that $N > 2/\delta$. Thus for all $n \geq N$ we have that $f_n(x) = 0$ for any $x \in [1/2 - \delta, 1/2 - \delta/2]$. Thus

$$\int_0^{1/2} |f_n(x) - f(x)|^2 \mathrm{d}x \geq \delta/4.$$

So f_n cannot converge to f and we can conclude that the norm $\|\cdot\|_2$ is not complete on $C([0,1])$ and we do not have a Hilbert space.

Remark. *It is in fact possible to extend the space $C([0,1])$ so we do get a Hilbert space. We could first extend to Riemann integrable functions, however this is still not complete and $\|\cdot\|_2$ is no longer a norm. To get round this we need to extend to the space of functions where $|f|^2$ is Lebesgue integrable. To get round the problem that $\|\cdot\|_2$ is not a norm we need to identify functions which only disagree on measure zero sets (if you're not familiar with measure don't worry). We then have $L^2([0,1])$ which is a Hilbert space.*

Closest point property

We now turn to an important property of Hilbert spaces which is not true in general for Banach spaces.

Example. Work in $\mathbb{R}^2$ and consider the subspace $A = \{(x, x) : x \in \mathbb{R}\}$. Take the point $(-1, 1)$. If we work with the Euclidean distance (coming from the ℓ^2 norm) then we can see that there is an unique point on A closer to $(-1, 1)$ than any other point, namely $(0, 0)$. However if we work with with the distance from the ℓ^1 norm then we can see this is no longer true and there are uncountably many closest points.

Definition 2.5. *Let V be a vector space over a field $\mathbb{F}$ (where $\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$). We say that $A \subseteq V$ is convex if for any two point $x, y \in A$ we have that if $0 < \lambda < 1$ then $\lambda x + (1 - \lambda)y \in A$.*

Theorem 2.6 (Parallelogram rule)**.** *Let V be an inner product space, for each $v \in V$ let $\|v\| = \sqrt{\langle v, v\rangle}$, and $x, y \in V$. We have that*

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

Proof. See exercise sheet 1 question 4. □

We combine this result which exploits the inner product with the completeness of Hilbert spaces to show that all Hilbert spaces have the closest point property.

Theorem 2.7 (Closest point property)**.** *Let H be a Hilbert space and let $A \subset H$ be a closed convex subset of H. We then have that for any $y \in H$ there exists $a \in A$ an unique closest point to y, i.e. an unique $a \in A$ such that*

$$\|y - a\| = \inf_{x\in A}\{\|y - x\|\}.$$

Proof. Let A be a closed convex set and let $y \in H$. We let $M = \inf_{x\in A}\{\|y - x\|\}$ and x_n be a sequence in A such that $\lim_{n\to\infty}\|y - x_n\| = M$. We now use the parallelogram rule to show that x_n must be a Cauchy sequence. Let $\epsilon > 0$ and choose N such that for all $n \geq N$ we have that $|\|y - x_n\|^2 - M^2| \leq \epsilon$. Now take $n, m \geq N$ and apply the parallelogram rule to $y - x_n$ and $y - x_m$. This gives

$$\|y - x_n - (y - x_m)\|^2 + \|y - x_n + y - x_m\|^2 = 2\|y - x_n\|^2 + 2\|y - x_m\|^2.$$

Rearranging this gives that

$$\begin{aligned}\|x_n - x_m\|^2 &= 2\|y - x_n\|^2 + 2\|y - x_m\|^2 - \|y - x_n + y - x_m\|^2 \\ &\leq 4M^2 + 4\epsilon - 4\|y - (x_n + x_m)/2\|^2.\end{aligned}$$

Since A is convex it follows that $(x_n + x_m)/2 \in A$ and thus $\|y - (x_n + x_m)/2\|^2 \geq M^2$. Thus

$$\|x_n - x_m\| \leq 4\epsilon.$$

So x_n is a Cauchy sequence, and since H is a Hilbert space and A is closed there must exist $x \in A$ such that $\lim_{n\to\infty} x_n = x$ and thus $\|x - y\| \geq M$. Moreover for any $n \in \mathbb{N}$ we have that

$$\|x - y\| \leq \|x - x_n\| + \|x_n - y\|$$

and we can let $n \to \infty$ to obtain $\|x - y\| \leq M$ and thus $\|x - y\| = M$.

Finally we need to show uniqueness we let $z \in A$ satisfy that $\|z - y\| = \|x - y\|$. By convexity we must have that $(z + x)/2 \in A$ and thus by the parallelogram rule

$$\|z - y - (x - y)\|^2 + \|z - y + (x - y)\|^2 = 2\|x - y\|^2 + 2\|z - y\|^2.$$

Rearranging this and using that $\|(z + x)/2 - y\| \geq M$ gives that

$$\begin{aligned} \|z - x\|^2 &= 4M^2 - \|z - y + (x - y)\|^2 \\ &\leq 4M^2 - 4\|(z + x)/2 - y\|^2 \leq 0. \end{aligned}$$

Thus $\|z - x\| = 0$ and so $z = x$. □