3 Orthogonality and Fourier series

We now turn to the concept of orthogonality which is a key concept in inner product spaces and Hilbert spaces. We start with some basic definitions.

Definition 3.1. *Let V be an inner product space and $A \subset V$. We say that A is an orthogonal subset if for all $x, y \in A$ we have that $x \neq y$ implies $\langle x, y \rangle = 0$.*

Definition 3.2. *Let V be an inner product space. We say that $A \subset V$ is an orthonormal subset of V if A is an orthogonal subset and for all $x \in A$ we have that $\|x\| = 1$. If A can be indexed by the natural numbers we write $A = \{e_1, e_2, \ldots\} = \{e_n\}_{n\in\mathbb{N}}$ and call it an orthonormal sequence.*

Example. Working in ℓ^2 if we let e_n be the sequence with nth term 1 and all other terms 0. It is easy to see that this gives an orthonormal sequence. In $C((-\pi, \pi))$ the space of complex valued continuous functions $f : (-\pi, \pi) \to \mathbb{C}$ with inner product $\langle f, g \rangle = \int_{-\pi}^{\pi} f\overline{g}\mathrm{d}x$ we let $e_n(x) = (2\pi)^{-1/2}e^{inx}$ and we can see that this gives an orthonormal sequence ($\{e_n\}_{n\in\mathbb{Z}}$).

Definition 3.3. *Let H be a Hilbert space and $\{e_n\}_{n\in\mathbb{N}}$ be an orthonormal sequence in H. For any $x \in H$ we say that $\langle x, e_n \rangle$ is the nth Fourier coefficient of x with respect to $\{e_n\}_{n\in N}$. We say that the formal series $\sum_{n=1}^{\infty} \langle x, e_n \rangle e_n$ is the Fourier series of x with respect to $\{e_n\}_{n\in\mathbb{N}}$.*

We now use the properties of Hilbert space to address the question of when we can give proper meaning to the Fourier series. However our first results just use the inner product structure and do not require completeness.

Theorem 3.4. *Let V be an inner product space and $\{x_1, \ldots, x_n\}$ be an orthogonal subset of V. We have that*

$$\|\sum_{k=1}^{n} x_k\|^2 = \sum_{k=1}^{n} \|x_k\|^2.$$

Proof. This can be done by simply expanding the left hand side as an inner product (the case $n = 2$ was exercise sheet 1 question 5). □

Note that with $n = 2$ in Euclidean space this is Pythagoras' Theorem.

Theorem 3.5. *Let V be an inner product space, $\{e_1, \ldots, e_n\}$ be an orthogonal subset of V and $A = span\{e_1, \ldots, e_n\}$. For any $x \in V$ we have that the closest point in A to x is $\sum_{k=1}^{n} \langle x, e_k \rangle e_k$ and*

$$\|x - \sum_{k=1}^{n} \langle x, e_k \rangle e_k\|^2 = \|x\|^2 - \sum_{k=1}^{n} |\langle x, e_k \rangle|^2.$$

Proof. Let $x \in V$ and $\lambda_1, \ldots, \lambda_n \in F$. We can calculate

$$\begin{aligned}
\|x - \sum_{k=1}^{n} \lambda_k e_k\|^2 &= \langle x - \sum_{k=1}^{n} \lambda_k e_k, x - \sum_{k=1}^{n} \lambda_k e_k \rangle \\
&= \langle x, x\rangle - \langle \sum_{k=1}^{n} \lambda_k e_k, x\rangle - \langle x, \sum_{k=1}^{n} \lambda_k e_k \rangle + \sum_{k=1}^{n} \lambda_k \overline{\lambda_k} \\
&= \|x\|^2 - \sum_{k=1}^{n} (\lambda_k \overline{\langle x, e_k\rangle} + \overline{\lambda_k}\langle x, e_k\rangle - \lambda_k \overline{\lambda_k}) \\
&= \|x\|^2 - \sum_{k=1}^{n} |\langle x, e_k\rangle|^2 + \sum_{k=1}^{n} (\langle x, e_k\rangle \overline{\langle x, e_k\rangle} - \lambda_k \overline{\langle x, e_k\rangle} - \overline{\lambda_k}\langle x, e_k\rangle - \lambda_k \overline{\lambda_k}) \\
&= \|x\|^2 - \sum_{k=1}^{n} |\langle x, e_k\rangle|^2 + \sum_{k=1}^{n} (\langle x, e_k\rangle - \lambda_k)(\overline{\langle x, e_k\rangle - \lambda_k}) \\
&= \|x\|^2 - \sum_{k=1}^{n} |\langle x, e_k\rangle|^2 + \sum_{k=1}^{n} |\langle x, e_k\rangle - \lambda_k|^2.
\end{aligned}$$

We can clearly see that this sum is minimised if we take $\langle x, e_j\rangle = \lambda_j$ for all $1 \leq j \leq n$ and the result follows. □

Corollary 3.6. *Let V be an inner product space, $\{e_1, \ldots, e_n\}$ be an orthogonal subset of V and $A = span\{e_1, \ldots, e_n\}$. If $x \in A$ then $x = \sum_{k=1}^{n} \langle x, e_k\rangle e_k$.*

Theorem 3.5 allows us to deduce the following inequality which will be a key tool when we look at the convergenc eof Fourier series in Hilbert space.

Corollary 3.7 (Bessel's inequality)**.** *Let V be an inner product space and let $\{e_n\}_{n\in\mathbb{N}}$ be an orthonormal sequence in V. We have that*

$$\sum_{n=1}^{\infty} |\langle x, e_n\rangle|^2 \leq \|x\|^2.$$

Proof. Fix $n \in \mathbb{N}$ and note that for any $x \in V$ we have that by Theorem 3.5

$$0 \leq \|x - \sum_{k=1}^{n} \langle x, e_k\rangle e_k\|^2 = \|x\|^2 - \sum_{k=1}^{n} |\langle x, e_k\rangle|^2.$$

Thus

$$\sum_{k=1}^{n} |\langle x, e_k\rangle|^2 \leq \|x\|^2$$

and since this holds for all $n \in \mathbb{N}$ we can conclude that

$$\sum_{n=1}^{\infty} |\langle x, e_n\rangle|^2 \leq \|x\|^2.$$

□

We can now exploit these results in the situation where we are working in a Hilbert space. First of all we need to introduce some standard notaion.

Definition 3.8. *Let V be a normed space with norm $\|\cdot\|$. For a seuqence $\{x_n\}_N \in \mathbb{N}$ in V and $x \in V$ we write $x = \sum_{n=1}^{\infty} x_n$ if and only if*

$$\lim_{k\to\infty} \|\sum_{n=1}^{k} x_n - x\| = 0.$$

If such an x exists we say that the series $\sum_{n=1}^{\infty} x_n$ is convergent.

In general it is hard to say when a series in a normed space is convergent (in fact this is true even if we are working in Banach or Hilbert spaces). However if we are working in Hilbert spaces with orthogonal sequences then there is a simple characterisation of when series are convergent.

Theorem 3.9. *Let H be an Hilbert space and let $\{e_n\}_{n\in\mathbb{N}}$ be an orthonormal sequence in H. For a seuqence $\{\lambda_n\}_{n\in\mathbb{N}}$ in $\mathbb{F}$ we have that $\sum_{n=1}^{\infty} \lambda_n e_n$ is convergent if and only if $\sum_{n=1}^{\infty} |\lambda_n|^2 < \infty$.*

Proof. We first suppose that $\sum_{n=1}^{\infty} \lambda_n e_n$ is convergent and let $x = \sum_{n=1}^{\infty} \lambda_n e_n$. In that case, using the continuity of the inner product (see exercise sheet 1 question 6) for $k \in \mathbb{N}$ we can write

$$\langle x, e_k\rangle = \langle \lim_{j\to\infty} \sum_{n=1}^{j} \lambda_n e_n, e_k\rangle = \lim_{j\to\infty} \langle \sum_{n=1}^{j} \lambda_n e_n, e_k\rangle = \lambda_k.$$

Thus by Bessel's inequality

$$\sum_{k=1}^{\infty} |\lambda_k|^2 = \sum_{k=1}^{\infty} |\langle x, e_k\rangle|^2 \le \|x\|^2.$$

To prove the converse suppose that $\sum_{n=1}^{\infty} |\lambda_n|^2 < \infty$ and fix $n, p \in \mathbb{N}$. By Pythagoras's Theorem (Theorem 3.4) we have that

$$\|\sum_{k=1}^{n+p} \lambda_k e_k - \sum_{k=1}^{n} \lambda_k e_k\|^2 = \sum_{k=n+1}^{n+p} \|\lambda_k e_k\|^2 = \sum_{k=n+1}^{n+p} |\lambda_k|^2.$$

By fixing $\epsilon > 0$ and choosing N such that $\sum_{k=N}^{\infty} |\lambda_k|^2 < \epsilon^2$ we can deduce that $\sum_{k=1}^{n} \lambda_k e_k$ is a Cauchy sequence and is thus convergent. □

Corollary 3.10. *In a Hilbert space, Fourier series $\sum_{n=1}^{\infty} \langle x, e_n\rangle\, e_n$ converge.*

We can now answer the question of what properties we need for an orthonormal sequence, $\{e_n\}_{n\in\mathbb{N}}$ to be able to write $x = \sum_{n=1}^{\infty}\langle x, e_n\rangle e_n$ for all $x \in H$. By the previous result combined with Bessel's Theorem we know that $\sum_{n=1}^{\infty}\langle x, e_n\rangle e_n$ is convergent. So we can let $y = x - \sum_{n=1}^{\infty}\langle x, e_n\rangle e_n$ and note that again using the continuity of the inner product we have that for all $j \in \mathbb{N}$

$$\begin{aligned}\langle y, e_j\rangle &= \langle x, e_j\rangle - \langle\sum_{n=1}^{\infty}\langle x, e_n\rangle e_n, e_j\rangle \\ &= \langle x, e_j\rangle - \langle x, e_j\rangle = 0.\end{aligned}$$

So if we have that if the only element $z \in H$ for which $\langle z, e_j\rangle = 0$ for all $j \in \mathbb{N}$ is 0 then we have that $x = \sum_{n=1}^{\infty}\langle x, e_n\rangle x$ for all $x \in H$.

Definition 3.11. *Let H be a Hilbert space and $A \subset H$. If A is an orthonormal subset of H and if $x \in H$ and $\langle x, a\rangle = 0$ for all $a \in A$ implies that $x = 0$ then A is a complete orthonormal subset of H. If A can be indexed by $\mathbb{N}$ then we cal A a complete orthonormal sequence.*

Theorem 3.12. *Let H be an Hilbert space and $\{e_n\}_{n\in\mathbb{N}}$ be a complete orthonormal sequence. We then have that for all $x \in H$, $x = \sum_{n=1}^{\infty}\langle x, e_n\rangle x$ and $\|x\|^2 = \sum_{n=1}^{\infty}|\langle x, e_n\rangle|^2$.*

Proof. We have already proved the first part of the Thereom with the argument proceeding the definition of a complete orthonormal subsequence. For the second part fix $x \in H$ and note that

$$\begin{aligned}\|x\|^2 &= \langle\sum_{n=1}^{\infty}\langle x, e_n\rangle e_n, \sum_{n=1}^{\infty}\langle x, e_n\rangle e_n\rangle \\ &= \lim_{k\to\infty}\sum_{n=1}^{k}|\langle x, e_n\rangle|^2 = \sum_{n=1}^{\infty}|\langle x, e_n\rangle|^2.\end{aligned}$$

□

We can also use similar arguments to characterise when an orthonormal sequence is a complete orthonormal sequence.

Theorem 3.13. *Let H be an Hilbert space and let $\{e_n\}_{n\in\mathbb{N}}$ be an orthonormal sequence in H. The following three statements are equivalent*

1. *$\{e_n\}_{n\in\mathbb{N}}$ is a complete orthonormal seuqence.*
2. *$H = \overline{span(\{e_n : n \in \mathbb{N}\})}$.*
3. *For all $x \in H$ we have that $\|x\|^2 = \sum_{n=1}^{\infty}|\langle x, e_n\rangle|^2$.*

Proof. The fact that statemnt 1 implies 2 and 3 follow from the previous theorem. To see that statement 2 implies statement 1 suppose that $H =$

$\overline{\mathrm{span}(\{e_n : n \in \mathbb{N}\})}$ and let $x \in H$ satisfy that $\langle x, e_n \rangle = 0$ for all $n \in \mathbb{N}$. We consider the subspace of H

$$A = \{y \in H : \langle x, y \rangle = 0\}.$$

By the continuity of the inner product it follows that this is closed and we know that $e_n \in H$ for all $n \in \mathbb{N}$. Thus $A = H$ and in particular $x \in H$. Thus $\langle x, x \rangle = 0$ and we have that $x = 0$. Thus $\{e_n\}_{n \in \mathbb{N}}$ is complete. Finally to show that statement 3 implies statement 1 suppose that $\|x\|^2 = \sum_{n=1}^{\infty} |\langle x, e_n \rangle|^2$ for all $x \in H$ and $y \in H$ satisfies $\langle y, e_n \rangle = 0$ for all $n \in \mathbb{N}$. We then have that $\|y\|^2 = 0$ and thus $y = 0$. So $\{e_n\}_{n \in \mathbb{N}}$ must be complete. $\square$

Example.

1. In ℓ^2 if we let e_n be the seuqence with nth term 1 and all other terms 0 then we can easilty see that we have a complete orthonormal sequence.

2. In $L^2([-\pi, \pi])$ it is possible to show that $e_n = (2\pi)^{-1/2} e^{inx}$ for $n \in \mathbb{Z}$ is a complete orthonormal sequence (see chapter 5 of Introduction to Hilbert spaces by Young). This is a possible topic for the level M project part of the course, particularly with you knowing some measure theory. This allows us deduce results about convergence for Fourier series in L^2. For $f \in L^2([-\pi, \pi])$, we set $c_n = (2\pi)^{-1/2} \int_{-\pi}^{\pi} f(x) e^{-inx} dx$. Then as $N \to \infty$,
$$\sum_{n=-N}^{N} c_n e^{inx} \to f \quad \text{in } L^2,$$
and
$$\sum_{n=-\infty}^{\infty} |c_n|^2 = \int_{-\pi}^{\pi} |f(x)|^2 dx.$$
In particular, we note that trigonometric polynomials are dense in $L^2([-\pi, \pi])$.

Orthogonal complements

We start by recalling a notion from linear algebra

Definition 3.14. *Let V be a vector space and M, N be subspaces of V. We write $V = M \oplus N$ if for all $x \in V$ there exist $y \in M$ and $z \in N$ such that $x = y + z$ and $M \cap N = \{0\}$.*

If we are working in a finite dimensional vector space V and we have a subspace N it is easy to find a supspace M such that $V = N \oplus M$ (take a basis for N and extend it to a basis for V). It turns out that if we take a general Banach space and a closed subspace we cannot always do this (see exercise sheet). However we can do this for Hilbert spaces using orthogonality.

Definition 3.15. *Let V be an inner product space and $M \subset V$. We define $M^\perp$ the orthogonal complement of M by*

$$M^\perp = \{x \in V : \langle x, y\rangle = 0 \text{ for all } y \in M\}.$$

Lemma 3.16. *Let H be a Hilbert space and $M \subset H$. $M^\perp$ is a closed subspace of H.*

Proof. See exercise sheet. □

It turns out in general inner product spaces there is a simple way of characterising the orthogonal complement for subspaces.

Lemma 3.17. *Let V be an inner product space and M a subspace of V. We have that $x \in M^\perp$ if and only if*

$$\|x\| \leq \|m - x\| \text{ for all } m \in M.$$

Proof. Firstly suppose that $x \in M^\perp$. Fix $m \in M$ and note that

$$\|x - m\|^2 = \langle x - m, x - m\rangle = \|x\|^2 + \|m\|^2 \geq \|x\|^2.$$

Thus $\|x\| \leq \|x - m\|$.

Conversely suppose that $x \in V$ and

$$\|x\| \leq \|m - x\| \text{ for all } m \in M.$$

Thus for all $\lambda \in F$ we have that $\|x\| \leq \|x - \lambda m\|$. This means that

$$0 \leq |\lambda|^2\|m\|^2 - 2\mathrm{Re}(\overline{\lambda}\langle x, m\rangle).$$

This holds for all λ so we can take $\overline{\lambda} = tu$ where $t \in \mathbb{R}$ is positive and u satisfies that $u\langle x, m\rangle = |\langle x, m\rangle|$. This means that

$$2t|\langle x, m\rangle| \leq t^2\|m\|$$

and thus

$$\langle x, m\rangle \leq t/2\|m\|.$$

Since this holds for all $t > 0$ we must have that $\langle x, m\rangle = 0$. □

Theorem 3.18. *Let H be an Hilbert space and M be a closed subspace of H. We have that $H = M \oplus M^\perp$.*

Proof. Firstly if $x \in M \cap M^\perp$ we have that $\langle x, x\rangle = 0$ and so we can deduce that $x = 0$ and $M \cap M^\perp = \{0\}$.

Now let $z \in H$ and let $x \in M$ be the unique closest point in M to z (M is closed and convex). In other words $\|z - x\| \leq \|z - m\|$ for all $m \in M$. Now let $y = z - x$ and $m \in M$.

$$\|m - y\| = \|m + x - z\| = \|z - (x + m)\| \geq \|z - x\| = \|y\|$$

and thus $y \in M^\perp$. □

From Theorem 3.18, every $x \in H$ can be written uniquely as $x = m + m^{\perp}$ with $m \in M$ and $m^{\perp} \in M^{\perp}$. This defines the orthogonal projection map $P_M : x \mapsto m$.