4 Linear operators and linear functionals

The next section is devoted to studying linear operators between normed spaces.

Definition 4.1. *Let V and W be normed spaces over a field $\mathbb{F}$. We say that $T : V \to W$ is a linear operator if T is linear (that is, $T(x+y) = T(x)+T(y)$ for all $x, y \in V$ and $T(\lambda x) = \lambda T(x)$ for all $x \in V$ and $\lambda \in \mathbb{F}$).*

Definition 4.2. *Let V and W be normed spaces. We say that a linear operator $T : V \to W$ is bounded if there exists $M > 0$ such that $\|x\| \leq 1$ implies $\|T(x)\| \leq M$. In such cases we define the operator norm by*

$$\|T\| = \sup\{\|T(x)\| : x \in V \text{ and } \|x\| \leq 1\}.$$

Example.

1. Not all operators are bounded. Let $V = C([0, 1])$ with respect to the norm $\|f\| = \left(\int_0^1 |f(x)|^2 \mathrm{d}x\right)^{1/2}$. Consider the linear operator $T : V \to \mathbb{C}$ given by $T(f) = f(0)$. We can see that this operator is unbounded by defining functions where f_n where $f_n(0) = n$ but $\int_0^1 |f_n|^2 \mathrm{d}x = 1$.

2. On the otherhand if we define $T : \ell^\infty \to \ell^\infty$ by $T(\{x_n\}_{n\in\mathbb{N}}) = T(\{y_n\})$ where $y_n = x_{n+1}$ we can see that we have a bounded operator. Since if $\|x\|_\infty \leq 1$ we have that $\|T(x)\| \leq 1$. Moreover if we take $x = (0, 1, 0, 0, \ldots)$ we can see that $\|x\|_\infty = 1$ and $\|T(x)\|_\infty = 1$ and so $\|T\| = 1$.

Our first key result related bounded operators to continuous operators.

Theorem 4.3. *Let V and W be normed spaces and $T : V \to W$ a linear operator. Then the following statements are equivalent*

1. *T is bounded*
2. *T is continuous*
3. *T is continuous at 0.*

Proof. We first show 1 implies 2. Suppose that T is bounded and let $x \in V$. Let $\epsilon > 0$ and fix $\delta = \epsilon/\|T\|$. For y such that $\|x - y\| \leq \delta$ we have that

$$\|T(x - y)\| \leq \|T\|\|x - y\| \leq \epsilon.$$

The fact that 2 implies 3 is obvious. Finally suppose that T is continuous at 0 and choose δ such that if $y \in V$ and $\|y\| \leq \delta$, then $\|T(x)\| \leq 1$. Now if $\|x\| \leq 1$, then $\|\delta x\| \leq \delta$ and so $\|T(\delta x)\| \leq 1$. Thus $\|T(x)\| \leq \delta^{-1}$. So we can conclude that T is bounded. $\square$

Definition 4.4. *Let V and W be normed spaces. We let $B(V,W)$ be the space of Bounded linear operators $T : V \to W$.*

Theorem 4.5. *Let V be a normed space and W a Banach space. The space $B(V,W)$ is a Banach space with respect to the operator norm.*

Proof. It's a routine exercise to show that $B(V,W)$ is a normed space. Now let $\{T_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $B(V,W)$. Let $x \in V$ and consider the sequence $T_n(x)$. Let $\epsilon > 0$ and choose N such that for all $n, m \geq N$ we have that $\|T_n - T_m\| \leq \epsilon$. We then have that

$$\|T_n(x) - T_m(x)\| = \|(T_n - T_m)(x)\| \leq \|x\|\|T_n - T_m\| \leq \|x\|\epsilon.$$

So $T_n(x)$ is a Cauchy sequence in W, and since W is Banach, we may define $T(x) = \lim_{n\to\infty} T_n(x)$.

We now need to show that $T \in B(V,W)$. Well T is linear since each T_n is linear. So let $\epsilon > 0$ and choose $N \in \mathbb{N}$ such that for $n, m \geq N$ we have that $\|T_n - T_m\| \leq \epsilon$. Take $x \in V$ such that $\|x\| \leq 1$ and note that for $n, m \geq N$, we have

$$\|T(x) - T_n(x)\| \leq \|T(x) - T_m(x)\| + \|T_m(x) - T_n(x)\| \leq \|T(x) - T_m(x)\| + \epsilon.$$

Since this holds for all $m \geq N$ and $\lim_{m\to\infty}\|T(x) - T_m(x)\| = 0$ we have that $\|T_n(x) - T(x)\| \leq \epsilon$ for all $n \geq N$. Thus $\|T(x)\| \leq \|T_N(x)\| + \epsilon$ and so T is bounded. We can also conclude that for $n \geq N$, $\|T_n - T\| \leq \epsilon$ and so T_n converge to T in $B(V,W)$. Hence, $B(V,W)$ is a Banach space. □

Unless it is otherwise specified when we talk about $B(V,W)$ we mean the normed space where the norm is the operator norm.

Linear functionals and Dual spaces

We now look at a special class of linear operators whose range is the field $\mathbb{F}$.

Definition 4.6. *If V is a normed space over $\mathbb{F}$ and $T : V \to \mathbb{F}$ is a linear operator, then we call T a linear functional on V.*

Definition 4.7. *Let V be a normed space over $\mathbb{F}$. We denote $B(V,\mathbb{F}) = V^*$ (recall that $B(V,\mathbb{F})$ is the the space of bounded operators for V to $\mathbb{F}$). We can V^* the dual space of V.*

Corollary 4.8 (Corollary to Theorem 4.5)**.** *For any normed space V we have that V^* is a Banach space.*

In the case of ℓ^p space where $p \neq \infty$ we can now determine exactly what the dual space is.

Example. Consider the space ℓ^1. We can find a norm preserving linear bijection (isomorphism for Banach spaces) between the dual space $(\ell^1)^*$ and ℓ^∞. For $x = (x_n)_{n\in\mathbb{N}}$ define $f_x : \ell^1 \to \mathbb{F}$ by $f_x(y) = \sum_{n=1}^\infty x_n y_n$. We can see that f_x is linear and for $y \in \ell^1$, $|f_x(y)| \leq \|x\|_\infty \|y\|_1$, so f_x is a bounded linear functional. We can define $T : \ell^\infty \to (\ell^1)^*$ by $T(x) = f_x$ and easily see that it is a injective linear map.

To show that T is surjective let $f \in (\ell^1)^*$ and consider the elements $e_n \in \ell^1$ with nth term 1 and all other terms 0. We have that $c = (f(e_n))_{n\geq 1} \in \ell^\infty$ and for any $y = (y_n)_{n\in\mathbb{N}} \in \ell^1$ note that $y = \lim_{k\to\infty} \sum_{n=1}^k y_n e_n$. So by the continuity of f,

$$f(y) = \lim_{n\to\infty} \sum_{n=1}^{k} f(e_n) y_n = \sum_{n=1}^{\infty} f(e_n) y_n = f_c(y).$$

Finally we need to show T preserves norms. To do this fix $x \in \ell^\infty$ and note we have already shown that $\|f_x\| \leq \|x\|_\infty$. Now let $\epsilon > 0$ and n such that $|x_n| \geq \|x\|_\infty - \epsilon$. We have that

$$|f_x(e_n)| \geq \|x\|_\infty - \epsilon$$

and since $\|e_n\|_1 = 1$ we can conclude that $\|f_x\| = \|x\|_\infty$. Thus for all $x \in \ell^\infty$, $\|T(x)\| = \|x\|_\infty$.

For Hilbert spaces, it turns out that Hilbert spaces can be identified with their own dual spaces.

Theorem 4.9 (Riesz-Frechet)**.** *Let H be a Hilbert space. For any $f \in H^*$ there exists an unique $y \in H$ such that $f(x) = \langle x, y\rangle$ for all $x \in H$ and $\|f\| = \|y\|$.*

Proof. The case $f = 0$ is obvious. Let's assume that $f \neq 0$.

We first address the existence of such a y. Let $f \in H'$. Let

$$A = \ker(f) = \{x \in H : f(x) = 0\}$$

and note that A is a closed subspace of H and $A \neq H$. So $A^\perp$ is a closed subspace of H which does not just contain 0 (since $H = A \oplus A^\perp$). Thus we can now pick $z \in A^\perp$ such that $f(z) = 1$. For $x \in H$ we can write $x = (x - f(x)z) + f(x)z$ and we'll have that $x - f(x)z \in A$ and $f(x)z \in A^\perp$. Thus for all $x \in H$ we have

$$\langle x, z\rangle = \langle x - f(x), z\rangle + \langle f(x)z, z\rangle = f(x)\|z\|^2.$$

We can now let $y = z/\|z\|^2$ to get for all $x \in H$ $f(x) = \langle x, y\rangle$.

To show that such a y is uniquely determined we simply note that if $\langle x, y\rangle = \langle x, z\rangle$ for all $x \in H$, then $\langle x, y - z\rangle = 0$ for all $x \in H$ and so $y = z$.

Finally we need to show $\|f\| = \|y\|$ which is again trivial for $y = 0$. For $y \neq 0$ we have by Cauchy-Schwarz that

$$|\langle x, y\rangle| \leq \|x\|\|y\|$$

and so $\|f\| \leq \|y\|$. Moreover

$$|f(y/\|y\|)| = |\langle y/\|y\|, y\rangle| = \|y\|$$

and so $\|f\| = y$. □

For general Banach spaces or normed spaces it is much harder to characterise the dual space (even ℓ^∞ is more difficult than these examples). Later on in the unit you will see the Hahn-Banach theorem which gives a very useful way of defining linear functionals (for instance it gives a way of showing that there are nonzero functional on any non-trivial normed space, surprisingly this is a non-trivial fact).

Inverse operators, adjoints and unitary operators

We now turn to the question of when operators have inverses and then move on to the notion of adjoint operators and finally the notion of a unitary operator. In finite dimension vector spaces we can write a linear operator as a matrix and use the determinant to determine whether it is invertible or not. In general normed spaces things are quite a bit more complicated.

Definition 4.10. *Let V be a normed space. Let $I_v \in B(V, V)$ be the identity operator on V where*

$$I_V(x) = x \text{ for all } x \in V.$$

Definition 4.11. *Let V and W be normed space. We say that $T \in B(V, W)$ is invertible if there exists $S \in B(W, V)$ such that*

$$TS = I_W \text{ and } ST = I_V$$

and we call $S = T^{-1}$ the inverse of T.

If V is a finite dimensional vector space and $T \in B(V) = B(V, V)$ then the following are all equivalent to T being invertible

1. T is injective
2. T is surjective
3. There exists $S \in B(V)$ such that $ST = I_V$.
4. There exists $T \in B(V)$ such that $TS = I_V$.

For normed spaces none of these conditions all equivalent to an operator being invertible.

Example. *Let $V = \ell^\infty$ and $T : \ell^\infty \to \ell^\infty$ be the shift map defined by*

$$T((x_1, x_2, \ldots,)) = (x_2, x_3, \ldots))$$

and $S : \ell^\infty \to \ell^\infty$ be defined by

$$S((x_1, x_2, \ldots)) = (0, x_1, x_2, \ldots).$$

We can see that T is surjective, but not injective and that $TS = I$ but not $ST = I$.

However for Banach spaces there is a criteria which can be used to find examples of invertible operators.

Theorem 4.12. *Let V be a Banach space and $A \in B(V)$ with $\|A\| < 1$. We have that $I - A$ is invertible and*

$$(I - A)^{-1} = \sum_{n=0}^{\infty} A^n.$$

(note that we take $A^0 = I$.)

Proof. To start we show that $\sum_{n=0}^{\infty} A^n \in B(V)$. Let $A_k = \sum_{n=0}^{k} A^n$. We will show that A_k is a Cauchy sequence in $B(V)$. Let $n, m \geq N$ and let $x \in V$ such that $\|x\| \leq 1$. We have that

$$\|A_n(x) - A_m(x)\| = \|\sum_{k=n+1}^{m} A^k(x)\| \leq \sum_{k=n+1}^{m} \|A^k\| \leq \sum_{k=n+1}^{\infty} \|A^k\| \leq \sum_{k=n+1}^{\infty} \|A\|^k \frac{\|A\|^{n+1}}{1 - \|A\|}.$$

So we can see that A_k is a Cauchy sequence and so we can let $T = \lim_{k\to\infty} A_k$ and note that for all $x \in V$ $T(x) = \lim_{k\to\infty} A_k(x)$. We now need to show $T(I - A) = (I - A)T = I$. Let $x \in V$ we have that

$$T(I-A)(x) = T(x) - TA(x) = \lim_{k\to\infty} A_k(x) - A_k A(x) = \lim_{k\to\infty} x - A^{k+1}(x) = x$$

and we can also show $(I - A)T(x) = x$. □

Corollary 4.13. *Let V be a Banach space. The set of invertible operators is open in $B(V)$.*

Proof. Note that if $A, B \in B(V)$ are invertible then AB is invertible and that I is invertible so there is always at least one invertible element in $B(V)$. We need to show that for any invertible operator A there exists $\delta > 0$ such that if $S \in B(V)$ and $\|A - S\| < \delta$ then S is invertible. Let $A \in B(V)$ be invertible with inverse A^{-1}. Consider

$$X = \{T \in B(V) : \|A - T\| < (\|A^{-1}\|)^{-1}\}.$$

For $S \in X$ we have that

$$\|(S - A)A^{-1}\| \leq \|S - A\|\|A^{-1}\| < 1.$$

Thus by Theorem 4.12 we know that $I + (S - A)A^{-1} = SA^{-1}$ is invertible. Thus $S = SA^{-1}A$ is invertible. □

Adjoint, self-adjoint and unitary operators

Definition 4.14. *Let V_1 and V_2 be normed spaces. For $T \in B(V_1, V_2)$, we define the map $T^* : V_2^* \to V_1^*$ by $f \mapsto f \circ T$, $f \in V_2^*$.*

Theorem 4.15. *The map $T^* \in B(V_2^*, V_1^*)$ and $\|T^*\| \leq \|T\|$.*

Proof. We first show that T^* is linear. For $f, g \in V_2^*$ and $\mu, \nu \in \mathbb{F}$, the functional $T^*(\mu f + \nu g)$ is defined by $x \mapsto (\mu f + \nu g)(Tx) = \mu f(Tx) + \nu g(Tx) = (\mu T^*(f) + \nu T^*(g))(x)$. Hence, T^* is linear. Moreover, for $f \in V_2^*$ and $x \in V_1$ with $\|x\| \leq 1$, $|T^*(f)(x)| = |f(Tx)| \leq \|f\|\|Tx\| \leq \|f\|\|T\|$. This shows that $\|T^*\| \leq \|T\|$. □

In the context of Hilbert spaces, the adjoint operator can be written more explicitly using the Riesz-Frechet theorem. Let H_1 and H_2 be Hilbert spaces and $T \in B(H_1, H_2)$. For each $y \in H_2$ we can define a linear functional $f : H_1 \to \mathbb{F}$ by $f_y(x) = \langle T(x), y\rangle$ where the inner product is taken in H_2. By Cauchy-Schwartz it follows that $f_y \in H_1^*$ and by the Riesz-Frechet Theorem there exists a unique $z \in H_1$ such that $\langle T(x), y\rangle = f_y(x) = \langle x, z\rangle$. This defines the adjoint operator $T^* : H_2 \to H_1$.

Example. Let $T : \ell^2 \to \ell^2$ be the left shift map $T((x_1, x_2, x_3, \ldots)) = (x_2, x_3, \ldots)$. For $y \in \ell^2$ we have that

$$\langle T(x), y\rangle = \sum_{n=1}^{\infty} x_{n+1}\overline{y_n} = \sum_{n=1}^{\infty} x_n\overline{z_n}$$

where $z_n = (0, y_1, y_2, \ldots)$. So $T^* : \ell^2 \to \ell^2$ is the right shift where

$$T^*(x_1, x_2, \ldots) = (x_2, x_3, \ldots).$$

Theorem 4.16. *Let H_1 and H_2 be Hilbert spaces and $T \in B(H_1, H_2)$. We have that*

1. $(T^*)^* = T$

2. $\|T^*\| = \|T\|$

Proof. Let $x \in H_2$ and $y \in H_1$. We have that

$$\langle x, T(y) \rangle = \overline{\langle T(y), x \rangle} = \overline{\langle y, T^*(x) \rangle} = \langle T^*(x), y \rangle = \langle x, (T^*)^*(y) \rangle.$$

This holds for all $x \in H_2$ and so $T = (T^*)^*$. Since $\|T^*\| \leq \|T\|$, we can use this to deduce that $\|T\| = \|T^*\|$. □

See Theorem 3.9-4 in Introductory Functional Analysis with Applications by Kreyszig for more properties of the adjoint.

Definition 4.17. *Let H be a Hilbert space and let $T \in B(H)$. We call T self-adjoint (often called Hermitian) if and only if $T = T^*$. We call T unitary if $T^* = T^{-1}$.*

Example. Take $H = \ell^2$, $a = (a_n)_{n \geq 1} \in \ell^\infty$ and $T : \ell^2 \to \ell^2$ be given by $T(x) = (a_n x)_{n \geq 1}$. For $x, y \in \ell^2$ we have that

$$\langle T(x), y \rangle = \sum_{n=1}^{\infty} (a_n x_n) \overline{y_n} = \sum_{n=1}^{\infty} x_n \overline{\overline{a_n} y_n}.$$

So we can see that T is self-adjoint if and only if all $a_n \in \mathbb{R}$, and T is unitary if and only if $|a_n| = 1$ for all n.