8 Baire Category Theorem and Uniform Boundedness Principle

8.1 Baire's Category Theorem

Validity of many results in analysis depends on the completeness property. This property addresses the inadequacy of the system of rational numbers. The manner in which completeness of a metric space X is typically exploited is that the intersection of nested closed balls

$$\bar{B}(x_1, r_1) \supset \bar{B}(x_2, r_2) \supset \cdots \supset \bar{B}(x_n, r_n) \supset \cdots$$

with $r_n \to 0$ is non-trivial. To see that this is the case, we observe that the sequence x_n satisfies for $m \geq n$,

$$d(x_n, x_m) \leq r_n. \tag{1}$$

This implies that this sequence is Cauchy, and $x_n \to x$ for some $x \in X$. It follows from (1) that $d(x_n, x) \leq r_n$, so that the limit x gives a point in the intersection $\cap_{n\geq 1} \bar{B}(x_n, r_n)$.

A variation of this argument also gives:

Theorem 8.1 (Baire's Category Theorem)**.** *Let X be a complete metric space and U_n, $n \geq 1$, a collection of open dense subsets of X. Then the intersection $\cap_{n\geq 1} U_n$ is dense in X.*

Proof. It is sufficient to show that $\cap_{n\geq 1} U_n$ intersects non-trivially any ball $B(x_0, r_0)$ with radius $r_0 \in (0, 1)$. We construct a sequence of nested open balls $B(x_n, r_n)$ inductively. Given a ball $B(x_{n-1}, r_{n-1})$, we observe that $U_n \cap B(x_{n-1}, r_{n-1})$ is non-empty and open. So that we can choose x_n, r_n so that $\bar{B}(x_n, r_n) \subset U_n \cap B(x_{n-1}, r_{n-1})$. We may also choose r_n's so that $r_n \to 0$. Then by our previous discussion, there exists $x \in \cap_{n\geq 1} \bar{B}(x_n, r_n)$. We obtain that for every n,

$$x \in \bar{B}(x_n, r_n) \subset U_n \cap B(x_0, r_0),$$

which completes the proof. □

It is natural to think about open dense sets (and their countable intersections) as "large" subsets of X, and their complements as "small" subsets of X. This motivates the following definition.

Definition 8.2. Let X be a metric space and M a subset of X.

- M is called *rare* (or *nowhere dense*) if its closure $\overline{M}$ has no interior points.

- M is called *meager* (or of *first category*) in X if it is a countable union of rare sets. Otherwise M is called *nonmeager* (or of *second category*).

Example. $\mathbb{Q}$ is not rare in $\mathbb{R}$ because its closure is all of $\mathbb{R}$. But $\mathbb{Q} \times \{0\}$ *is* rare in $\mathbb{R}^2$, its closure being $\mathbb{R} \times \{0\}$, which has no interior points as a subset of $\mathbb{R}^2$. $\mathbb{Q}^n$ is meager in $\mathbb{R}^n$, being a countable union of points.

The Baire Category Theorem can be also restated as follows:

Theorem 8.3 (Baire's Category Theorem)**.** *A nonempty complete metric space X is nonmeager.*

Proof. Suppose that in contrary

$$X = \bigcup_{n \geq 1} M_n \tag{2}$$

where M_k's are closed sets with empty interior. Then the sets $X \backslash M_n$ are open and dense. Hence, by Theorem 8.1,

$$\bigcap_{n \geq 1} U_n \neq \emptyset.$$

This implies that the complement

$$X \backslash \left(\bigcap_{n \geq 1} U_n \right) = \bigcup_{n \geq 1} M_n \neq X,$$

which gives a contradiction. Hence, X cannot be meager. □

8.2 Application: nowhere differentiable functions

The Baire Category Theorem is a very useful tool for showing existence of objects with peculiar properties. We demonstrate how this works by showing that there is continuous function which is not differentiable at even a single point.

Theorem 8.4. *The subset*

$$D = \{f \in C[a,b] : \ f \text{ is differentiable at some } x \in [a,b]\}$$

is a meager subset of $C[a,b]$.

Since the space $C[a,b]$ is complete, it follows that $D \subsetneq C[a,b]$, and there exist functions which are nowhere differentiable. In some sense, Theorem 8.4 tells us that "most" continuous functions are nowhere differentiable.

Proof. For $n, m \geq 1$, consider the sets

$$A_{n,m} = \left\{ f \in C[a,b] : \ \exists x : \ \left| \frac{f(t) - f(x)}{t - x} \right| \leq n \text{ for all } 0 < |x - t| < \tfrac{1}{m} \right\}.$$

Recall that f is differentiable at x if the limit

$$f'(x) = \lim_{t \to x} \frac{f(t) - f(x)}{t - x}$$

exists. So that it is clear that

$$D \subset \bigcup_{n,m \geq 1} A_{n,m}.$$

It remains to show that the sets $A_{m,n}$ are rare.

First we show that $A_{n,m}$ is closed. Suppose that $f_i \to f$ in $C[a,b]$ and all $f_i \in D$. Then there exists $x_i \in [a,b]$ such that

$$\left| \frac{f_i(t) - f_i(x_i)}{t - x_i} \right| \leq n \text{ for all } 0 < |x_i - t| < \tfrac{1}{m}.$$

Passing to a subsequence, we may assume that $x_i \to x$ for some $x \in [a,b]$. Then if $0 < |x - t| < \frac{1}{m}$, then for sufficiently large n, we also have $0 < |x_i - t| < \frac{1}{m}$, and

$$\left| \frac{f(t) - f(x)}{t - x} \right| = \lim_{i \to \infty} \left| \frac{f_i(t) - f_i(x_i)}{t - x_i} \right| \leq n.$$

This proves that the sets $A_{n,m}$ are closed.

Now we show that the sets $A_{n,m}$ have empty interiors. Suppose that $A_{m,n}$ contains an open ball $B(f, \epsilon)$ for some $f \in C[a,b]$. Using uniform continuity, one shows that the subspace of piecewise linear continuous functions is dense in $C[a,b]$. Hence, without loss of generality we may assume that f is piecewise linear. For piecewise linear continuous functions, the one-sided derivatives $f'_+(x)$ and $f'_-(x)$ always exist, and they are uniformly bounded: for some $M > 0$,

$$|f'_\pm(x)| \leq M \quad \text{for all } x \in [a,b].$$

We observe that every function

$$g = f + \frac{\epsilon}{2}\phi \quad \text{with } \phi \in C[a,b] \text{ such that } \|\phi\|_\infty \leq 1$$

belongs to $B(f, \epsilon) \subset A_{n,m}$. In particular, it follows that there exists $x_0 \in [a,b]$ such that

$$|g'_\pm(x_0)| \leq n. \tag{3}$$

For every $K > 0$, one can construct (exercise) a piecewise linear function ϕ such that $\|\phi\|_\infty \leq 1$ and $|\phi'_\pm(x)| > K$. Then for all $x \in [a,b]$,

$$|g'_\pm(x)| \geq \frac{\epsilon}{2}|\phi'_\pm(x)| - |f'_\pm(x)| \geq \frac{\epsilon}{2}K - M.$$

Taking $K = K(\epsilon, M)$ sufficiently large, we obtain $|g'_\pm(x)| > n$ for all x, but this contradicts (3). Hence, the sets $A_{n,m}$ have empty interiors. □

8.3 Uniform boundedness principle

The uniform boundedness principle answers the question of whether a *"pointwise bounded"* sequence of bounded linear operators must also be *"uniformly bounded."*

Theorem 8.5 (Uniform Boundedness Principle; Banach–Steinhaus)**.** *Let X be a Banach space and Y be a normed space. Suppose that the sequence $T_n \in B(X,Y)$ of bounded linear operators has the property that for every $x \in X$, the sequence $T_n(x) \in Y$ is bounded. Then the sequence of norms $\|T_n\|$ is bounded.*

Let us illustrate the Uniform Boundedness Principle by an example.

Example 8.6. Let

$$X = \left\{ p(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \cdots + \alpha_d x^d \mid \alpha_i \in \mathbb{F}, d \in \mathbb{N} \cup \{0\} \right\},$$

be the space of polynomials equipped with the norm $\|p\| = \max_i |\alpha_i|$. We readily see that this turns X into a normed space. We give an example of a sequence of linear maps $T_n : X \to \mathbb{F}$ which are pointwise bounded but not uniformly bounded. Let

$$T_n(p) = \alpha_0 + \cdots + \alpha_{n-1}.$$

We observe that

$$|T_n(p)| = |\alpha_0 + \alpha_1 + \cdots + \alpha_{n-1}| \leq |\alpha_0| + \cdots + |\alpha_{n-1}| \leq n\|p\|,$$

so that $\|T_n\| \leq n$. In fact, this estimate can be improved for polynomials p of degree d as follows:

$$\begin{aligned} |T_n(p)| &= |\alpha_0 + \alpha_1 + \cdots + \alpha_{n-1}| \leq |\alpha_0 + \alpha_1 + \cdots + \alpha_d| \\ &\leq |\alpha_0| + |\alpha_1| + \cdots + |\alpha_d| \leq d\,\|p\|, \end{aligned}$$

This shows that the sequence $(T_n(p))_{n \geq 1}$ is bounded for every p.

However, we claim that $\|T_n\| = n \to \infty$. Indeed, for

$$p_n(x) = 1 + x + x^2 + \cdots + x^{n-1},$$

we have $\|p_n\| = 1$, and $\|T_n(p_n)\| = n$, which implies the claim.

In this example, the sequence $(T_n(p))_{n \geq 1}$ is pointwise bounded, but not uniformly bounded, contrary to the conclusion of the Uniform Boundedness Principle. The Uniform Boundedness Principle fails here because the space X is not complete.

We give a concrete application of the Uniform Boundedness Principle.

Example 8.7. Suppose we have a sequence of complex numbers $x = (x_n)_{n\geq 1}$ with the property that whenever $y = (y_n)_{n\geq 1}$ is a sequence satisfing $y_n \to 0$, we have that the sum $\sum_{n=1}^{\infty} x_n y_n$ converges. Show that $\sum_{n=1}^{\infty} |x_n|$ converges.

The first step is of course to translate this problem into a more convenient functional analytic setting. The condition that the sequence y converges to 0 is simply the statement that y is a member of the space

$$c_0 = \{y \in \ell^\infty : y_n \to 0 \text{ as } n \to \infty\}.$$

One can show that c_0 is a Banach space when equipped with the norm $\|\cdot\|_\infty$. We leave this as an exercise. We are required to show that $x \in \ell^1$.

Since we would like to use the Uniform Boundedness Principle in some way, let us start by finding a sequence $T_n \in B(c_0, Y)$ where Y is some normed space. One natural choice is to let

$$T_n(y) = \sum_{i=1}^{n} x_i\, y_i \quad \text{where} \quad y = (y_1, y_2, \dots) \in c_0$$

be the truncated sum which we have assumed converges. It is clear that $T_n \in B(c_0, \mathbb{C}) = (c_0)^*$. Indeed, the calculation

$$|T_n(y)| = \left|\sum_{i=1}^{n} x_i\, y_i\right| \leq \sum_{i=1}^{n} |x_i\, y_i| \leq \left(\sum_{i=1}^{n} |x_i|\right) \|y\|_\infty$$

shows that T_n's are bounded operators with $\|T_n\| \leq \sum_{i=1}^{n} |x_i|$. In fact, we can do better: the assumption that $\sum_i x_i\, y_i$ converges for any $y \in c_0$ implies that for any $y \in c_0$, the sequence $(T_n(y))_{n\geq 1}$ is convergent, hence bounded. So $(T_n)_{n\geq 1}$ is a *pointwise bounded* sequence of operators. Therefore, by the Uniform Boundedness Principle, it is uniformly bounded, meaning that there is some $M > 0$ such that $\|T_n\| \leq M$ for all $n \in \mathbb{N}$.

We claim that $\|T_n\| = \sum_{i=1}^{n} |x_i|$. Without loss of generality, we may assume that $x \neq 0$. We consider $y^{(n)} = (y_i^{(n)})_{i\geq 1} \in c_0$ defined by

$$y_i^{(n)} = \begin{cases} \bar{x}_i^{(n)}/|x_i^{(n)}| & \text{if } i \leq n \text{ and } x_i \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$T_n(y^{(n)}) = \sum_{i=1}^{n} |x_i|,$$

and $\|y^{(n)}\|_\infty = 1$ unless $y^{(n)} = 0$. This implies that $\|T_n\| \geq \sum_{i=1}^{n} |x_i|$. The opposite inequality has already been proven above. We conclude that $\sum_{i=1}^{n} |x_i| \leq M$ for all n, and $\sum_{i=1}^{\infty} |x_i|$ converges.

8.4 Proof of the Uniform Boundedness Principle

The proof of the Uniform Boundedness Principle is an application of Baire's Category Theorem. Let us define the sets

$$M_k = \{x \in X : \|T_n(x)\| \leq k \text{ for all } n\}, \quad k \geq 1.$$

Since T_n's are continuous, these sets are closed. Since for every $x \in X$, the sequence $T_n(x)$ is bounded, we have $x \in M_k$ for sufficiently large k. Therefore,

$$X = \bigcup_{k \geq 1} M_k.$$

Baire's category theorem now guarantees that one of these closed sets contains an open ball, say $B(x_0, r) \subset M_{k_0}$. We therefore have that

$$\|T_n(x)\| \leq k_0 \quad \text{for any } x \in B(x_0, r) \text{ and } n \geq 1.$$

Now let $x \in X$, $x \neq 0$. Then the vector $z = x_0 + \frac{r}{2\|x\|} x$ belongs to $B(x_0, r)$ and $x = \frac{2\|x\|}{r}(z - x_0)$. Using this, we calculate

$$\begin{aligned} \|T_n(x)\| &= \frac{2\|x\|}{r} \|T_n(z) - T_n(x_0)\| \leq \frac{2\|x\|}{r} (\|T_n(z)\| + \|T_n(x_0)\|) \\ &\leq \frac{4k_0\|x\|}{r}. \end{aligned}$$

Hence, $\|T_n\| \leq \frac{4k_0}{r}$ for all $n \in \mathbb{N}$, proving that the sequence is uniformly bounded.