# 9 Modes of convergence

## 9.1 Weak convergence in normed spaces

We recall that the notion of convergence on a normed space $X$, which we used so far, is the convergence with respect to the norm on $X$: namely, for a sequence $(x_n)_{n\geq 1}$, we say that $x_n \to x$ if

$$\|x_n - x\| \to 0 \quad \text{as } n \to \infty. \tag{1}$$

However, in many cases, this notion does not capture the full information about behaviour of sequences. Moreover, we recall that according to the Bolzano–Weierstrass Theorem, a bounded sequence in $\mathbb{R}$ has a convergent subsequence. An analogue of this result fails for in infinite dimensional normed spaces if we only deal with convergence with respect to norm.

**Example 9.1.** Consider the sequence $e_n = (0, \ldots, 0, 1, 0, \ldots)$ in the space $\ell^2$. Then for $n \neq m$, $\|e_n - e_m\|_2 = \sqrt{2}$. So that we conclude that the sequence $(e_n)_{n\geq 1}$ is not Cauchy, so it cannot converge, and it even does not contain any convergent subsequence. Nonetheless, in some "weak" sense, we may say that this sequence "converges" to $(0, \ldots, 0, \ldots)$. For instance, for every $x = (x_n)_{n\geq 1} \in \ell^2$,

$$\langle x, e_n \rangle = x_n \to 0.$$

This example motivates the notion of weak convergence which we now introduce.

**Definition 9.2.** A sequence $(x_n)_{n\geq 1}$ in a normed space $X$ *converges weakly* to $x \in X$ if for every $f \in X^*$, we have that $f(x_n) \to f(x)$. We write

$$x_n \xrightarrow{w} x.$$

To emphasise the difference with the usual notion of convergence, if (1) hold, we say $(x_n)_{n\geq 1}$ *converges in norm* or *converges strongly*.

Returning to Example 9.1, we see that the sequence $(e_n)_{n\geq 1}$ converges weakly, but has no subsequences that converge strongly. Indeed, any $f \in (\ell^2)^*$ is of the form $f(y) = \langle y, x \rangle$ for some $x \in \ell^2$, and $\langle e_n, x \rangle = \bar{x}_n \to 0$.

Here is another example of a sequence which converges weakly, but not strongly.

**Example 9.3.** Let $X$ be the space of real-valued continuous functions with the max-norm, and

$$\phi_n(t) = \begin{cases} nt & \text{when } 0 \leq t \leq 1/n, \\ 2 - nt & \text{when } 1/n \leq t \leq 2/n, \\ 0 & \text{when } 2/n \leq t \leq 1. \end{cases}$$

We claim that $\phi_n \xrightarrow{w} 0$. Suppose that this is not the case. Then there exists $f \in X^*$ such that $f(\phi_n) \not\to 0$. Passing to a subsequence we may assume that $|f(\phi_{n_i})| \geq \delta$ for some fixed $\delta > 0$, and without loss of generality

$$f(\phi_{n_i}) \geq \delta.$$

Moreover, we may assume that the subsequence satisfies $n_{i+1} \geq 2n_i$. Let

$$\psi_N = \sum_{i=1}^{N} \phi_{n_i}$$

We have

$$\psi_N(t) \leq \sum_{i:\, n_i \leq 1/t} n_i t + \sum_{i:\, 1/t < n_i \leq 2/t} (2 - n_i t).$$

Let $k = \max\{i :\ n_i \leq 1/t\}$. Then for all $i \leq k$,

$$n_i \leq n_k / 2^{k-i} \leq 1/t \cdot 1/2^{k-i},$$

and the first sum satisfies

$$\leq \sum_{i \leq k} 1/2^{k-i} \leq 2.$$

To estimate the second sum we observe that because $n_{i+1} \geq 2n_i$, the inequality $1/t < n_i \leq 2/t$ may holds for at most one index $i$. Hence, the second sum is also bounded by 2. Hence,

$$\|\psi_N\|_\infty \leq 4 \quad \text{and} \quad |f(\psi_N)| \leq 4\|f\|.$$

On the other hand,

$$f(\psi_N) = \sum_{i=1}^{N} f(\phi_{n_i}) \geq N\delta \to \infty \quad \text{as } N \to \infty.$$

This gives a contradiction.

Since $\|\phi_n\|_\infty = 1$, $\phi_n \not\to 0$ strongly.

We derive some basic properties of weak convergence.

**Theorem 9.4.** *1. If $x_n \to x$, then $x_n \xrightarrow{w} x$.*

*2. Weak limits are unique.*

*3. If $(x_n)_{n \geq 1}$ is a weakly convergent sequence, then the sequence of norms $\|x_n\|$ is bounded.*

*Proof.* 1. This follows from the estimate $|f(x_n)-f(x)| \leq \|f\|\|x_n-x\|$ for every $f \in X^*$.

2. Suppose $x_n \xrightarrow{w} x$ and $x_n \xrightarrow{w} y$. Then for every $f \in X^*$, we have $f(x_n) \to f(x)$ and $f(x_n) \to f(y)$. Hence, it follows that $f(x) = f(y)$ for all $f \in X^*$. So that by a corollary of the Hahn–Banach Theorem, $x = y$.

3. We would like to apply the Uniform Boundedness Principle. For this we have to interpret $x_n$'s as maps on a suitable space. We consider the mapping $X \to (X^*)^*$ defined by $x \mapsto g_x$, where $g_x(f) = f(x)$ for any $f \in X^*$. Since
$$|g_x(f)| = |f(x)| \leq \|x\|\|f\|,$$
this indeed defines a bounded linear functional on $X^*$ with $\|g_x\| \leq \|x\|$. Moreover, by a corollary of Hahn–Banach Theorem, there exists $f \in X^*$ such that $f(x) = \|x\|$ and $\|f\| = 1$. This implies that
$$\|g_x\| = \sup\{|f(x)| : f \in X^*, \|f\| \leq 1\} \leq \|x\|.$$
Hence, $\|g_x\| = \|x\|$.

For short, put $g_{x_n} := g_n$. Since for every $f \in X^*$, the sequence $f(x_n)$ converges, it is bounded, that is, $|f(x_n)| \leq c_f$ where $c_f$ is some positive constant independent of $n$. Thus, $|g_n(f)| \leq c_f$ for all $n \in \mathbb{N}$. We recall that the dual space $X^*$ is always a Banach space. The Uniform Boundedness Principle tell us that the sequence $g_n$ is uniformly bounded, meaning $\|g_n\| \leq c$, where $c > 0$ is independent of $n$. This implies the claim.

□

The following theorem gives a convenient criterion for weak convergence.

**Theorem 9.5.** *A sequence $(x_n)_{n\geq 1}$ in a normed space $X$ converges weakly to $x$ provided that*

*(i) $\|x_n\|$ is uniformly bounded,*

*(ii) For every element $f$ in a dense subset $M \subset X^*$, we have $f(x_n) \to f(x)$.*

*Proof.* According to (i), $\|x_n\| < c$ and $\|x\| < c$ for some fixed $c > 0$.

We would like to show that for any $f \in X^*$, we have $f(x_n) \to f(x)$. Let $\epsilon > 0$ and $\{f_j\} \subset M \subset X^*$ be a sequence with $f_j \to f$ in $X^*$. We obtain
$$|f(x_n) - f(x)| \leq |f(x_n) - f_j(x_n)| + |f_j(x_n) - f_j(x)| + |f_j(x) - f(x)|. \quad (2)$$

Since $f_j \to f$, we can choose $j$ large enough that

$$\|f_j - f\| < \frac{\epsilon}{3c}.$$

Also, since $f_j(x_n) \to f_j(x)$, there is a number $N$ such that

$$|f_j(x_n) - f_j(x)| < \frac{\epsilon}{3}$$

whenever $n > N$. We can now bound (2)

$$\begin{aligned} |f(x_n) - f(x)| &\le |f(x_n) - f_j(x_n)| + |f_j(x_n) - f_j(x)| + |f_j(x) - f(x)| \\ &\le \|f - f_j\|\|x_n\| + |f_j(x_n) - f_j(x)| + \|f_j - f\|\|x\| \\ &\le \frac{\epsilon}{3c}c + \frac{\epsilon}{3} + \frac{\epsilon}{3c}c < \epsilon \end{aligned}$$

whenever $n > N$. This proves that $x_n \xrightarrow{w} x$. □

*Remark* 9.6 (weak convergence in Hilbert spaces). If $X$ is a Hilbert space then the Riesz–Frechét Theorem tells us that a sequence $(x_n)_{n\ge1} \subset X$ is weakly convergent to $x \in X$ if and only if $\langle x_n, z\rangle \to \langle x, z\rangle$ for all $z \in X$. Theorem 9.5 then tell us that we only need to check that $\langle x_n, v\rangle \to \langle x, v\rangle$ for elements $v$ of some basis of $X$.

## 9.2 Convergence of sequences of functionals

Now we consider convergence of a sequence of linear functional $f_n \in X^*$.

**Definition 9.7** (weak* convergence)**.** We say that a sequence $(f_n)_{n\ge1}$ *weak* converges* to $f \in X^*$ if for every $x \in X$ we have that $f_n(x) \to f(x)$. This is denoted by $f_n \xrightarrow{w^*} f$.

We note that since the dual space $X^*$ is also a normed space, it also makes sense to talk about strong and weak convergence in $X^*$. Namely:

- a sequence $f_n \in X^*$ *converges strongly* to $f$ if $\|f_n - f\| \to 0$.
- a sequence $f_n \in X^*$ *converges weakly* to $f$ if for every $g \in (X^*)^*$, we have $g(f_n) \to g(f)$.

In general, we have:

strong convergence ⇒ weak convergence ⇒ weak* convergence

To see that the second arrow is true, we note that every $x \in X$ defines an element $g_x \in (X^*)^*$ such that

$$g_x(f) = f(x). \tag{3}$$

This defines a map $X \to (X^*)^*$. We note that in general this map is not surjective and weak$^*$ convergence does not imply weak convergence.

We illustrate the notion of weak$^*$ convergence by some examples.

**Example 9.8.** Let $X = C[-1,1]$ be the space of continuous functions, and

$$\rho_n(t) = \begin{cases} n - n^2|t| & \text{when } -1/n \le t \le 1/n, \\ 0 & \text{otherwise.} \end{cases}$$

We consider the sequence functionals $f_n : X \to \mathbb{C}$ defined by

$$f_n(\phi) = \int_{-1}^{1} \phi(t)\rho_n(t)dt, \quad \phi \in C[-1,1].$$

We claim that $f_n$ weak$^*$ converges to $f_0$ defined by $f_0(\phi) = \phi(0)$. Indeed, using that $\int_{-1}^{1} \rho_n(t)\,dt = 1$, we obtain

$$\begin{aligned} |f_n(\phi) - f_0(\phi)| &= \left| \int_{-1}^{1} \phi(t)\rho_n(t)dt - \int_{-1}^{1} \phi(0)\rho_n(t)dt \right| \\ &\le \int_{-1}^{1} |\phi(t) - \phi(0)|\rho_n(t)\,dt \\ &= \int_{-1/n}^{1/n} |\phi(t) - \phi(0)|\rho_n(t)\,dt \\ &\le \max_{-1/n \le t \le 1/n} |\phi(t) - \phi(0)|. \end{aligned}$$

Hence, it follows from continuity of $\phi$ that $f_n(\phi) \to f_0(\phi)$. This proves that $f_n \xrightarrow{w^*} f_0$.

Although we will not prove it here, it is the case that $f_n \not\xrightarrow{w} f_0$

**Example 9.9.** Let

$$X = c_0 = \{x = (x_n)_{n \ge 1} \in \ell^\infty : x_n \to 0\}.$$

We have met this space in homeworks. We recall that $c_0^* \simeq \ell^1$. More explicitly, all elements of $c_0^*$ are of the form

$$f_y(x) = \sum_{n=1}^{\infty} x_n y_n, \quad x \in c_0$$

for some $y \in \ell^1$. We consider the sequence of linear functionals $f_n = f_{e_n}$. Then for every $x \in c_0$,

$$f_n(x) = x_n \to 0.$$

Hence, $f_n \xrightarrow{w^*} f$.

We also recall that $(\ell^1)^* \simeq \ell^\infty$, and all elements of $(\ell^1)^*$ are of the form

$$g_z(y) = \sum_{n=1}^{\infty} y_n z_n, \quad y \in \ell^1$$

for some $z \in \ell^\infty$. Since $g_z(e_n) = z_n \not\to 0$ in general. The sequence $f_n$ does not converge weakly to 0.

**Example 9.10.** If $H$ is a Hilbert space, the Riesz–Frechét Theorem tells us that $H^* \simeq H$, and the map $H \to (H^*)^*$ defined in (3) is an isomorphism. This implies that in Hilbert spaces weak and weak* convergences are the same.

The following result is a direct corollary of the Uniform Boundedness Principle.

**Theorem 9.11.** *If the sequence $(f_n)_{n\geq 1}$ in $X^*$ is weak* convergent, then the sequence $\|f_n\|$ is bounded.*

The following theorem is analogous to Theorem 9.5. It tell us a way to determine whether a given sequence in $X^*$ is weak* convergent, without having to check the defining condition on *all* the elements of $X$. Its proof also runs in parallel with the proof of Theorem 9.5.

**Theorem 9.12.** *A sequence $(f_n)_{n\geq 1}$ in $X^*$ is weak* convergent provided that*

*(i) The sequence $\|f_n\|$ is bounded.*

*(ii) The sequence $f_n(x)$ is Cauchy for every $x$ in a dense subset $M \subset X$.*

*Proof.* Fix $c > 0$ such that $\|f_n\| < c$ for all $n$. Now, let $x \in X$. We can find a sequence $x_j$ in $M$ such that $x_j \to x$. Let $\epsilon > 0$. We will show that we can make $|f_m(x) - f_n(x)| < \epsilon$ by taking large enough $m, n$. We carry out the first bound:

$$\begin{aligned} |f_m(x) - f_n(x)| &\leq |f_m(x) - f_m(x_j)| + |f_m(x_j) - f_n(x_j)| + |f_n(x_j) - f_n(x)| \\ &\leq \|f_m\|\|x - x_j\| + |f_m(x_j) - f_n(x_j)| + \|f_n\|\|x_j - x\|. \end{aligned}$$

Since $x_j \to x$, we can fix $j$ large enough that $\|x - x_j\| < \frac{\epsilon}{3c}$. Since the sequence $f_n(x_j)$ is Cauchy, there is a number $N$ such that whenever $m, n > N$, we have $|f_m(x_j) - f_n(x_j)| < \frac{\epsilon}{3}$. The above expression is now bounded by

$$< \frac{\epsilon}{3c}c + \frac{\epsilon}{3} + \frac{\epsilon}{3c}c = \epsilon,$$

which proves that the sequence $f_n(x)$ is indeed Cauchy, and so converges.

We define $f(x) = \lim_{n\to\infty} f_n(x)$ for all $x \in X$. Since $f_n$'s are linear, it follows that $f$ is also linear. For every $x \in X$, $|f_n(x)| \leq \|f_n\|\|x\| \leq c\|x\|$. Passing to the limit, we conclude that $|f(x)| \leq \|f_n\|\|x\| \leq c\|x\|$, so that $\|f\| \leq c$, and $f \in X^*$. We have $f_n \xrightarrow{w^*} f$. □

The following theorem is a generalisation of the Bolzano–Weierstrass theorem.

**Theorem 9.13** (weak* compactness)**.** *Suppose that a normed space $X$ contains a countable dense subset. Then every bounded sequence $f_n \in X^*$ contains a weak* convergent subsequence.*

*Proof.* Let $M = \{x_i\}_{i\geq 1}$ be a countable dense subset of $X$. Since $(f_n)_{n\geq 1}$ is a bounded sequence, $(f_n(x))_{n\geq 1}$ is also bounded for every $x \in X$. In particular, $(f_n(x_1))_{n\geq 1}$ is bounded, and by the Bolzano–Weierstrass Theorem, there is a subsequence $n_1(k)$ such that $(f_{n(k)}(x_1))_{k\geq 1}$ converges. Next, the sequence $(f_{n_1(k)}(x_2))_{k\geq 1}$ is bounded, and again by the Bolzano–Weierstrass Theorem, there is a subsequence $n_2(k)$ of the sequence $n_1(k)$ such that $(f_{n_2(k)}(x_2))_{k\geq 1}$ converges. Continuing this process, we construct subsequences $n_i(k)$ such that $(f_{n_i(k)}(x_i))_{k\geq 1}$ converges for all $i$. Now consider the sequence $n_k = n_k(k)$. It is a subsequence of each of the sequences $n_i(k)$. In particular, it follows that $(f_{n_k}(x_i))_{k\geq 1}$ converges for all $i$. Now we can just apply Theorem 9.12 to conclude that $f_{n_k}$ weak* converges. □

We note that an analogue of Theorem 9.13 for weakly convergent sequences is false (cf. Example 9.8).

## 9.3 Convergence of sequences of operators

Now we discuss convergence of bounded sequences of linear operators. There are three different types of convergence that arise naturally.

**Definition 9.14.** Let $X$ and $Y$ be normed spaces, and $T_n : X \to Y$ and $T : X \to Y$ are bounded linear operators. We say that:

- $T_n$ *converges uniformly* to $T$ (notation: $T_n \to T$) if

$$\|T_n - T\| \to 0.$$

- $T_n$ *converges strongly* to $T$ (notation: $T_n \xrightarrow{s} T$) if

$$T_n x \to Tx \quad \text{for all } x \in X.$$

- $T_n$ *converges weakly* to $T$ (notation: $T_n \xrightarrow{w} T$) if

$$f(T_n x) \to f(Tx) \quad \text{for all } x \in X \text{ and } f \in Y^*.$$

It is not hard to see that

uniform convergence $\Rightarrow$ strong convergence $\Rightarrow$ weak convergence

However, the converses are not true in general.

We illustrate the notions of convergence by several examples.

**Example 9.15.** Consider the sequence of operators $T_n : \ell^2 \to \ell^2$ defined by

$$T_n x = (x_{n+1}, x_{n+2}, \ldots), \quad x = (x_n)_{n \geq 1} \in \ell^2.$$

Clearly, $\|T_n x\| \leq \|x\|$, for all $x$, and for $k > n$, $\|T_n(e_k)\| = \|e_{k-n}\| = 1$. Hence, $\|T_n\| = 1$. In particular, it follows that $T_n \not\to 0$.

On the other hand, for every $x \in \ell^2$,

$$\|T_n x\| = \sqrt{\sum_{k=n+1}^{\infty} |x_k|^2} \to 0.$$

Hence, $T_n \xrightarrow{s} T$.

**Example 9.16.** Consider the sequence of operators $T_n : \ell^2 \to \ell^2$ defined by

$$T_n x = (0, \ldots, 0, x_1, x_2, \ldots), \quad x = (x_n)_{n \geq 1} \in \ell^2.$$

Then $\|T_n e_1\| = \|e_1\| = 1$. Hence, $T_n \not\xrightarrow{s} T$.

On the other hand, we claim that $T_n \xrightarrow{w} T$. We recall that $(\ell^2)^* \simeq \ell^2$, and every element in $(\ell^2)^*$ is given by

$$x \mapsto \langle x, y \rangle = \sum_{k=1}^{\infty} x_k y_k$$

for some $y = (y_n)_{n\geq 1} \in \ell^2$. For every $x, y \in \ell^2$,

$$|\langle T_n x, y\rangle| = \left|\sum_{k=1}^{\infty} x_k y_{k+n}\right| \leq \sqrt{\sum_{k=1}^{\infty} |x_k|^2}\sqrt{\sum_{k=n+1}^{\infty} |y_k|^2} \to 0.$$

This proves that $T_n \xrightarrow{w} T$.

**Example 9.17.** Let $X = C_c(\mathbb{R})$ be the space of continuous fuctions with compact support equipped with the max-norm. We consider the family of operators $T_a : X \to X$ defined by

$$T_a(\phi)(t) = \phi(t+a).$$

It is natural to expect that this family of operators depends continuously on $a$. Indeed, it follows from uniform continuity that for every $\phi \in X$,

$$\|T_a(\phi) - \phi\|_\infty = \max_t |\phi(t+a) - \phi(t)| \to 0$$

as $a \to \infty$. This shows that the map $a \to T_a$ is continuous at $a = 0$ with respect to the strong convergence.

On the other hand, for every fixed $a > 0$, we may consider a function $\phi \in C_c(\mathbb{R})$ such that $\{\phi \neq 0\} \subset [-a/3, a/3]$. Then it is easy to check that

$$\|T_a(\phi) - \phi\|_\infty = \|\phi\|_\infty.$$

So that $\|T_a - T_0\| = 1$, the map $a \to T_a$ is not continuous at $a = 0$ with respect to the uniform convergence.

We record some important properties of weak convergence.

**Theorem 9.18.** *If $T_n \in B(X, Y)$ is a weakly convergent sequence and $X$ is a Banach space, then the sequence of norms $\|T_n\|$ is bounded.*

*Proof.* If $T_n \xrightarrow{w} T$, then for every $x \in X$, $T_n x \xrightarrow{w} Tx$. Then by Theorem 9.4(iii), the sequence $(T_n x)_{n\geq 1}$ is bounded for every $x \in X$. Now the claim of the theorem follows from the Uniform Boundedness Principle. □

We also have an analogue of Theorem 9.5:

**Theorem 9.19.** *Let $T_n, T \in B(X, Y)$ where $X$ and $Y$ are normed spaces. The sequence $(T_n)_{n\geq 1}$ converges strongly to $T$ provided that*

*(i) $\|T_n\|$ is uniformly bounded,*

*(ii) For every element $x$ in a dense subset $M \subset X$, we have $Tx_n \to Tx$.*

This theorem is proved exactly as Theorem 9.5.