Homework 1

(1) Check that the hyperbolic area $\frac{dxdy}{y^2}$ is invariant under the action of $\mathrm{SL}_2(\mathbb{R})$.

(2) Let Γ be a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$. Show that if $F_1, F_2 \subset \mathbb{H}$ are fundamental domains for Γ, then $|F_1| = |F_2|$.

(3) Construct a fundamental domain for the group

$$\{\gamma \in \mathrm{SL}_2(\mathbb{Z}) : \gamma = I \mod 2\},$$

and show that $\mathrm{SL}_2(\mathbb{Z})/\langle \pm I \rangle$ is generated by the matrices

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.$$

(4) Show that the subgroup generated by

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 2\cos(\pi/n) \\ 0 & 1 \end{pmatrix}$$

with $n \geq 3$ is a lattice in $\mathrm{SL}_2(\mathbb{R})$.

(5) Show that every lattice subgroup is finitely generated. (Hint: Use the Siegel theorem.)

(6) Let $g_t : X \to X$ be a continuous flow on a topological space X. Assume that X has a countable basis for open sets, and X is equipped with a probability measure μ of full support (this means that $\mu(U) > 0$ for every open set $U \subset X$). Show that if g_t is mixing, then for a set of full measure in X, the orbit $\{g_t(x)\}_{t>0}$ is dense.

(7) Finish the proof of the corollary from the lecture that the periodic orbits are dense.

(8) Let $d \in \mathbb{N}$ and $(x, y) \in \mathbb{Z}^2$ be a solution of the Pell equation $x^2 - dy^2 = 1$. Show that every such solution gives rise to a periodic orbit of the flow $a_t = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}$ on $\mathrm{SL}_2(\mathbb{Z})\backslash\mathrm{SL}_2(\mathbb{R})$ with period $2\cosh^{-1}(x)$. Namely, construct $z \in \mathrm{SL}_2(\mathbb{Z})\backslash\mathrm{SL}_2(\mathbb{R})$ such that $za_{t_0} = z$ for $t_0 = \cosh^{-1}(x)$.

(9)
  (a) Show that for every $g \in \mathrm{SL}_2(\mathbb{Z})$ and $h \in \mathrm{SL}_2(\mathbb{Q})$, there exists $n \in \mathbb{N}$ such that $h^{-1}g^n h \in \mathrm{SL}_2(\mathbb{Z})$.
  (b) Show that if the orbit of $a_t = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}$ for the point $z_0 = \mathrm{SL}_2(\mathbb{Z})g_0 \in \mathrm{SL}_2(\mathbb{Z})\backslash\mathrm{SL}_2(\mathbb{R})$ is periodic, then so is the orbit for $z = \mathrm{SL}_2(\mathbb{Z})hg_0$ for every $h \in \mathrm{SL}_2(\mathbb{Q})$.
  (c) Deduce that the periodic orbits of the flow a_t in the space $\mathrm{SL}_2(\mathbb{Z})\backslash\mathrm{SL}_2(\mathbb{R})$ are dense.

(10) Let M be a hyperbolic surface. Show that if $x \in T^1(M)$ is periodic under the horocycle flow h_s, then the orbit $g_t(x)$ under the geodesic flow escapes every compact set as $t \to \infty$.