Homework 2

(1) Show that if $M = \Gamma\backslash\mathbb{H}$ is a compact hyperbolic surface, then Γ contains no unipotent elements.

(2) Deduce from mixing that the geodesic flow satisfies the mean ergodic theorem, namely, for every L^1 function,
$$\frac{1}{T}\int_0^T f(g_t(x))\,dt \to \int_X f$$
in L^1-norm as $T \to \infty$.

(3) Verify that the Laplace operator on $\mathbb{H}$ is given by
$$\Delta = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right).$$

(4) Determine all second order differential operators on $\mathbb{H}$ which commute with the action of $\mathrm{SL}_2(\mathbb{R})$.

(5) Let $M = \Gamma\backslash\mathbb{H}$ be a compact hyperbolic surface. Each geodesic loop ρ on M define two periodic geodesics ρ^+ and ρ^- on T^1M with opposite direction. The loop ρ is called *reciprocal* if $\rho^+ = \rho^-$. Prove that if Γ has no elements of finite order, then there are no reciprocal geodesics.

(6) Let $u(t,z)$ be a solution of the heat equation on a compact hyperbolic surface. Compute the limit of $u(t,z)$ as $t \to \infty$.

(7) Prove that the heat kernel on $\mathbb{H}$ that we constructed satisfies
$$\int_{\mathbb{H}} p_t(z,w)\,dm(w) = 1.$$
(Hint: to avoid lengthy computation, use the formula for the inverse Abel transform.)

(8) Prove that if two hyperbolic surfaces M_1 and M_2 have the same length spectrum $\mathcal{G}(M_1) = \mathcal{G}(M_2)$. Then the spectrum of the Laplace operators on M_1 and M_2 is also the same.

(9) Check that the Casimir operator Ω commutes with all other differential operators D_X and with all translation operators $\pi(g)$.

(10) Check that the Casimir operator on $\mathrm{SO}(2)$-invariant functions coincides with the Laplace operator.

(11) Prove that a derivative of a K-finite function is K-finite.

(12) Prove that the spaces $\mathcal{A}_n$ defined in the construction of the microlocal lift are pairwise orthogonal.