Homework 3

(1) Complete the estimates in the proof of the Weyl law (Lecture 6).

(2) Prove that for a function f on $\Gamma\backslash \mathrm{PGL}_2(\mathbb{R})$, $\Gamma = \mathrm{PGL}_2(\mathbb{Z})$, the Hecke operator T_p is given by

$$T_p f(\Gamma g) = \sum_{i=0}^{p} f(\Gamma \gamma_i g),$$

where

$$\gamma_0 = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \gamma_i = \begin{pmatrix} 1 & i-1 \\ 0 & p \end{pmatrix}, \quad i = 1, \dots p.$$

(Hint: describe all subspaces of $\mathbb{F}_p \oplus \mathbb{F}_p$.)

(3) Prove that the Hecke operators satisfy:

$$T_p^2 = T_{p^2} + (p+1)I,$$
$$T_p T_{p^{n-1}} = T_{p^n} + pI, \quad n \geq 2.$$

(4) Prove that the Hecke operators T_{p^n} are self-adjoint and commute with each other.

(5) Let $X_p = \mathrm{PGL}_2(\mathbb{Z}_p)\backslash \mathrm{PGL}_2(\mathbb{Q}_p)$ and $x \in \mathrm{PGL}_2(\mathbb{Z})\backslash \mathrm{PGL}_2(\mathbb{R})$. Show that the corresponding map

$$X_p \to \Delta_p(x)$$

to the Hecke tree at x is injective.

(6) Describe the orbits for the action of $\mathrm{SL}_2(\mathbb{Q}_p)$ on the tree $X_p = \mathrm{PGL}_2(\mathbb{Z}_p)\backslash \mathrm{PGL}_2(\mathbb{Q}_p)$.

In the following problems, $B_{n,\delta}(x)$ denotes the Bowen balls on $X = \Gamma\backslash \mathrm{SL}_2(\mathbb{R})$.

(7) Show that if $y \in B_{n,\delta}(x)$ and $z \in B_{n,\delta}(y)$, then $z \in B_{n,2\delta}(x)$.

(8) Let ν be the probability measure supported on a closed geodesic in $X = \Gamma\backslash \mathrm{SL}_2(\mathbb{R})$. Show that

$$\nu(B_{n,\delta}(x)) \not\to 0 \quad \text{as } n \to \infty.$$

(9) Let ν be the invariant measure on $X = \Gamma\backslash \mathrm{SL}_2(\mathbb{R})$. Show that

$$\nu(B_{n,\delta}(x)) \leq c(\delta, x) e^{-2n}.$$

(10) Let ρ be a geodesic line in $\mathbb{H}$ such that its endpoints are rational. Prove that its projection under $\pi : \mathbb{H} \to M = \mathrm{SL}_2(\mathbb{Z})\backslash \mathbb{H}$ is divergent (i.e., for every compact $\Omega \subset M$, the intersection $\pi(\rho) \cap \Omega$ is a bounded set of the line $\pi(\rho)$).