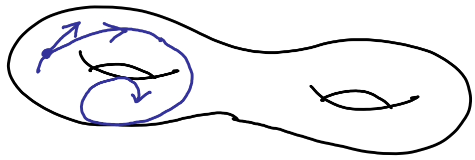
Lecture 1-2



Chaotic flows:

- geodesic flow,
- horocycle flow.

Eigenstates:

$$\Delta \varphi_\lambda = \lambda \varphi_\lambda$$

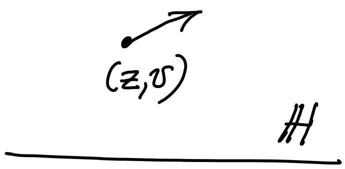
$\varphi_\lambda \approx ?$ as $\lambda \to \infty$.

Hyperbolic plane.

$$\mathbb{H} = \{ x+iy \in \mathbb{C} : y > 0 \}$$

$$T\mathbb{H} = \{ (z,v) : z \in \mathbb{H},\ v \in \mathbb{C} \}$$

└ tangent bundle of $\mathbb{H}$.



$$T\mathbb{H} = \bigsqcup_{z \in \mathbb{H}} T_z \mathbb{H}.$$

Hyperbolic metric: for $v \in T_z\mathbb{H}$, $\|v\|_z = \dfrac{\|v\|}{Im(z)}$.

For a paths $\varphi : [0,1] \to \mathbb{H}$,

the hyperbolic length: $L(\varphi) = \int_0^1 \|\varphi'(t)\|_{\varphi(t)}\, dt.$

For $z_1, z_2 \in \mathbb{H}$, $d(z_1,z_2) = \inf_{\varphi} L(\varphi)$
where φ runs over continuous piecewise differentiable curves with $\varphi(0)=z_1$, $\varphi(1)=z_2$.

Isometries: for $g=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$,

$$g \cdot z = \frac{az+b}{cz+d}.$$

Since $Im(g\cdot z) = \frac{Im(z)}{|cz+d|^2}$, $g\cdot \mathbb{H} \subset \mathbb{H}$.

Lem. $SL_2(\mathbb{R})$ acts transitively on $\mathbb{H}$,
$Stab(i) = SO(2) = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a^2+b^2=1 \right\}$.

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix} \cdot i = x+iy.$$

Lem. $SL_2(\mathbb{R})$ preserves the hyperbolic metric.

$$L(g\cdot\varphi) = \int_0^1 \|(Dg)_{\varphi(t)}\, \varphi'(t)\|_{g\cdot\varphi(t)}\, dt,$$

where $(Dg)_z v = \frac{v}{(cz+d)^2}$, so

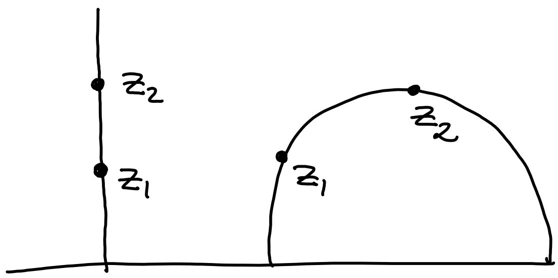
$$L(g\cdot\varphi) = \int_0^1 \frac{\|\varphi'(t)\|_{g\cdot\varphi(t)}}{|c\varphi(t)+d|^2}\, dt = \int_0^1 \frac{\|\varphi'(t)\|\, dt}{Im(g\cdot\varphi(t))\cdot|c\varphi(t)+d|^2}$$

$$= \int_0^1 \frac{\|\varphi'(t)\|}{Im(\varphi(t))}\, dt = L(\varphi).$$

Geodesics: A *geodesic* between $z_1, z_2 \in \mathbb{H}$ is a path between z_1 and z_2 such that $L(\varphi) = d(z_1, z_2)$.

Lem. The geodesic between $z_1, z_2 \in \mathbb{H}$ is either:

- vertical lines (if $Re(z_1) = Re(z_2)$),
- semicircle with the center on x-axis.



Γ Suppose that $Re(z_1) = Re(z_2)$.

Given a path $\varphi : [0,1] \to \mathbb{H}$ between z_1, z_2,

$$L(\varphi) = \int_0^1 \frac{\sqrt{\varphi_1'(t)^2 + \varphi_2'(t)^2}}{\varphi_2(t)} \geqslant \int_0^1 \frac{|\varphi_1'(t)|}{\varphi_2(t)}\, dt,$$

where "=" $\Leftrightarrow$ $\varphi_1' = 0$

Hence, the shortest path is the vertical line.

In general, we choose $g \in SL_2(\mathbb{R})$ so that

$$Re(g \cdot z_1) = Re(g \cdot z_2).$$

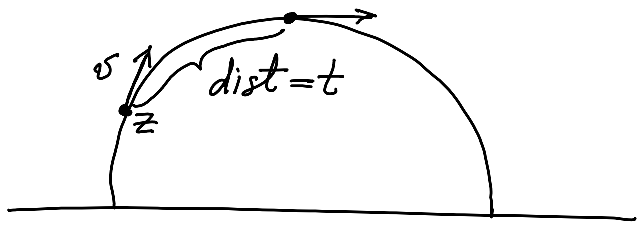
Then the vertical line L is the geodesic between $g \cdot z_1$ and $g \cdot z_2$, and $g^{-1} \cdot L$, which is a semi-circle, is the geodesic between z_1 and z_2.

Geodesic flow:

$$T^1\mathbb{H} = \{(z, v) : z \in \mathbb{H},\ v \in T_z\mathbb{H} : \|v\|_z = 1\}$$

unit tangent bundle

$g_t : T^1\mathbb{H} \to T^1\mathbb{H}$ - geodesic flow



$SL_2(\mathbb{R})$ acts on $T^1\mathbb{H}$:

$$(z, v) \xmapsto{g} (g \cdot z, (Dg)_z v) = \left(g \cdot z, \frac{v}{(cz+d)^2}\right).$$

$$\mathrm{Stab}((i, i)) = \{\pm I\}.$$

Hence, $T^1\mathbb{H} \simeq SL_2(\mathbb{R})/_{\{\pm I\}} = PSL_2(\mathbb{R})$.

Note that $d(i, a\cdot i) = \log a$ for $a > 1$,
(check)

so that $g_t \cdot (i,i) = (e^{t}\cdot i,\ e^{t}\cdot i)$

$$= \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} \cdot (i,i)$$

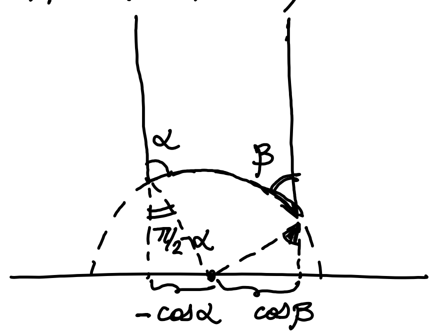
Using the identification $T^1\mathbb{H} \simeq PSL_2(\mathbb{R})$,

$$g_t : PSL_2(\mathbb{R}) \longrightarrow PSL_2(\mathbb{R})$$
$$g \longmapsto g \cdot \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}.$$

Hyperbolic area: $\dfrac{dx\,dy}{y^2}$ is invariant under $SL_2(\mathbb{R})$.

Lem. If T is a hyperbolic triangle with angles α, β, γ, then $|T| = \pi - \alpha - \beta - \gamma$.

Suppose that T has one of the vertices in $\mathbb{R} \cup \{\infty\}$. Applying $g \in SL_2(\mathbb{R})$, we may assume that $|v_1| = |v_2| = 1$, $v_3 = \infty$. Then

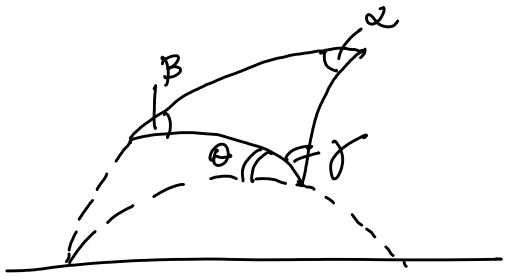


$$|T| = \int_{\cos(\pi-\alpha)}^{\cos\beta} \int_{\sqrt{1-x^2}}^{\infty} \frac{dx\,dy}{y^2}$$

$$= \pi - \alpha - \beta.$$

In general, we use the picture

$$|T| = \pi - \alpha - (\gamma + \theta) - (\pi - (\pi - \beta) - \theta)$$
$$= \pi - \alpha - \beta - \gamma.$$



<u>Cor.</u> If D is hyperbolic n-gon with angles $\alpha_1, \dots \alpha_n$, then $|D| = (n-2)\pi - \alpha_1 - \dots - \alpha_n$.

<u>Hyperbolic surfaces.</u>

Let Γ be a discrete subgroup of $SL_2(\mathbb{R})$.
We consider the space $X = \Gamma \backslash \mathbb{H}^2$.

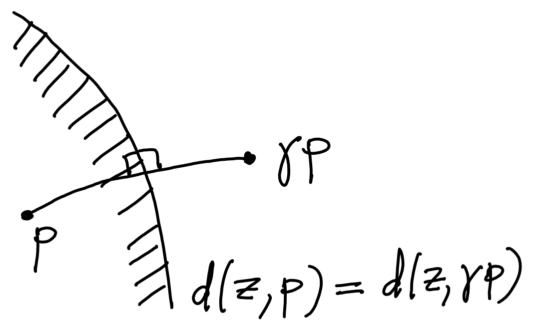
<u>Def</u> $F \subset \mathbb{H}^2$ is a <u>fundamental domain</u> if

1) $\bigcup_{\gamma \in \Gamma} \gamma F = \mathbb{H}^2$;
2) $\gamma_1 F^\circ \cap \gamma_2 F^\circ = \emptyset$ for $\gamma_1 \neq \gamma_2 \in \Gamma$.

Dirichlet fundamental domain:

Fix a point $p \in \mathbb{H}$, not fixed by $\gamma \in \Gamma \setminus \{e\}$.

$$D = \{ z \in \mathbb{H} : d(z,p) < d(z, \gamma p) \text{ for } \gamma \in \Gamma \setminus \{e\} \}.$$



D is convex

Prop. D is a fundamental domain.

Lem. The map $SL_2(\mathbb{R}) \to \mathbb{H} : g \mapsto g z_0$ is proper.

$\ulcorner$ Without loss of generality, $z_0 = i$.

$$g = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix} \underset{SO(2)}{\underset{\cap}{k}} \longmapsto x + iy \quad \lrcorner$$

Proof. By Lem., for every $z_0 \in \mathbb{H}$, the orbit Γz_0 is discrete. In particular,

$$\exists z \in \Gamma z_0 : \quad d(z,p) \le d(\gamma z_0, p) = d(z_0, \gamma^{-1} p)$$

for all $\gamma \in \Gamma$.

Then the geodesic segment $[p, z) \subset D$.

Hence, $z \in \overline{D}$, and $\Gamma \cdot \overline{D} = \mathbb{H}$.

Suppose that for $z_1, z_2 \in D$, $z_1 = \gamma z_2$ with $\gamma \neq e$.

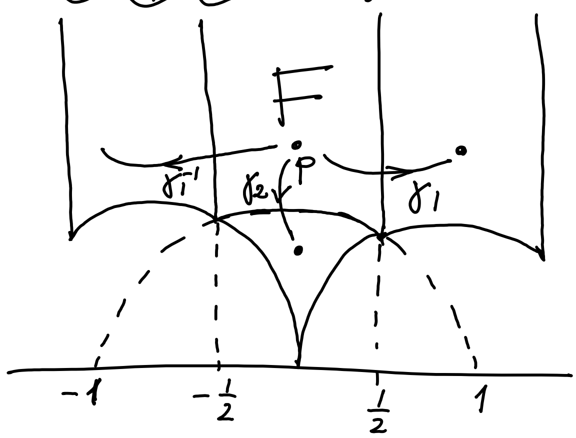
Then $d(z_1, p) < d(z_1, \gamma p) = d(\underbrace{\gamma^{-1} z_1}_{z_2}, p)$,

and similarly, $d(z_2, p) < d(z_1, p)$,

with is a contradiction.

Hence, $D \cap \gamma D = \emptyset$ for $\gamma \neq e$.

Example: 1) $\Gamma = SL_2(\mathbb{Z})$, $p = 2i$.



$\gamma_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $\gamma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \Gamma$.

It is clear that

$D \subset \underbrace{\{ |\mathrm{Re}(z)| < \tfrac{1}{2},\ |z| > 1 \}}_{F}$.

Suppose that $D \subsetneq F$.

Then $\exists z \in F$, $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, $\gamma \neq e$: $\gamma \cdot z \in F$.

We have $\mathrm{Im}(\gamma z) = \dfrac{\mathrm{Im}(z)}{|cz + d|^2}$

Since $|z|>1$ and $|Re(z)|<\frac{1}{2}$,

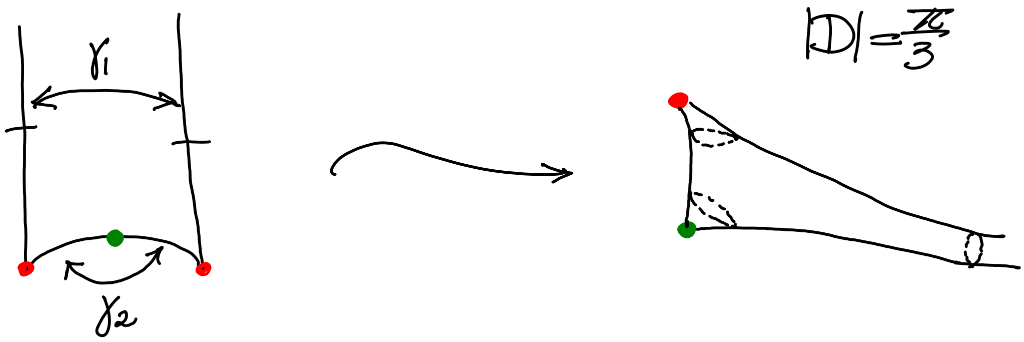
$$|cz+d|^2 = c^2|z|^2 + 2cd\,Re(z) + d^2 > c^2+d^2-|cd| = (|c|-|d|)^2+|cd|$$

Since $c,d\in\mathbb{Z}$ and $(c,d)\neq(0,0)$, $|cz+d|>1$.

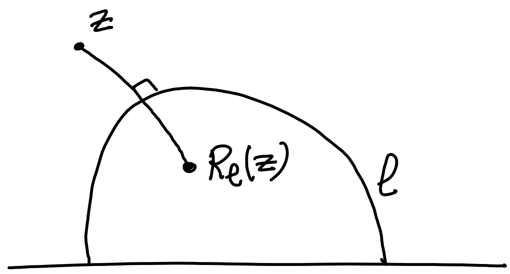
Hence, $Im(\gamma z)<Im(z)$.

Applying the same argument to $\gamma z \xrightarrow{\gamma^{-1}} z$,

we obtain $Im(z)<Im(\gamma z)$ — contradiction.



2) Triangle group.



For a geodesic ℓ, define the reflection map $z\longmapsto R_\ell(z)$.

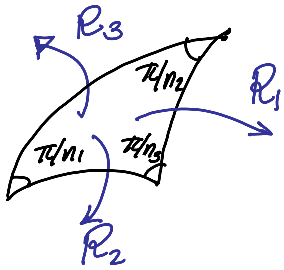
If $\ell_0 = y$-axis, $R_{\ell_0}(z) = -\bar{z}$.

In general take $g\in SL_2(\mathbb{R})$ such that $g\cdot\ell=\ell_0$. Then $R_\ell = g^{-1}\circ R_{\ell_0}\circ g$.

We note that:

- R_ℓ preserves the hyperbolic metric,
- $\langle PSL_2(\mathbb{R}), R_\ell : \ell\text{-geodesic} \rangle$ is index 2 supergroup of $PSL_2(\mathbb{R})$.

Take a triangle T with angles $\frac{\pi}{n_1}, \frac{\pi}{n_2}, \frac{\pi}{n_3}$, $n_i \in \mathbb{N}$.
(check: for every α, β, γ with $\alpha+\beta+\gamma<\pi$, there exists a triangle with angles α, β, γ)



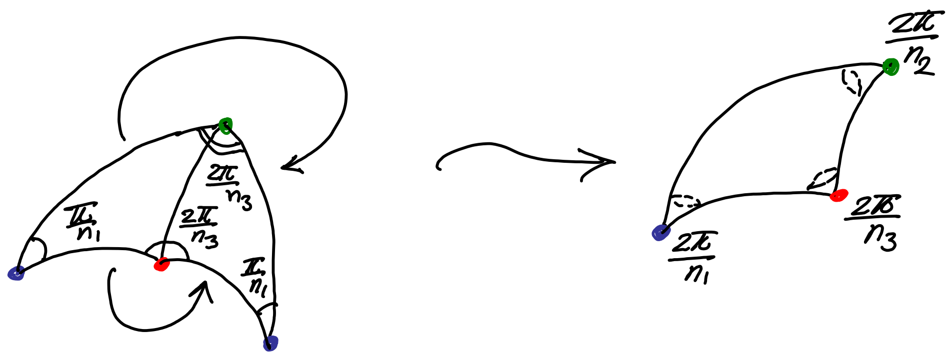
R_1, R_2, R_3 are reflections with respect to the sides.
Let $\Lambda = \langle R_1, R_2, R_3 \rangle$.

Since n_i's are integers,
$\lambda_1 T^\circ \cap \lambda_2 T^\circ \neq \emptyset \Rightarrow \lambda_1 T^\circ = \lambda_2 T^\circ \Rightarrow \lambda_1 = \lambda_2$.
Also, $\bigcup_{\lambda \in \Lambda} \lambda \overline{T} = \mathbb{H}$, i.e., T is a fundamental domain for Λ.

$\Lambda_0 = \{\text{even products of reflections}\} \underset{\text{index } 2}{\subset} \Lambda$.
Then $\Lambda_0 \subset PSL_2(\mathbb{R})$.

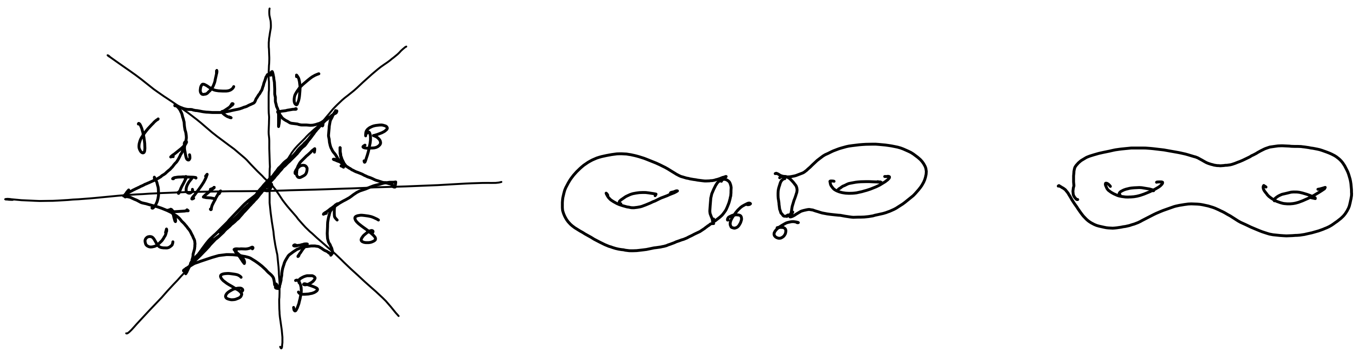
Prop. Λ_0 is discrete and cocompact.

Consider the map $p: SL_2(\mathbb{R}) \to \mathbb{H}: g \mapsto g \cdot i$.
Note that p is proper, and $p(g \cdot h) = g \cdot p(h)$.
If $F = T \cup R_1(T)$, then $\Lambda_0 \cdot F = \mathbb{H}$,
and $\Lambda_0 \cdot p^{-1}(F) = SL_2(\mathbb{R})$.
Hence, Λ_0 is cocompact.
If $K \subset SL_2(\mathbb{R})$ is compact, so is $p(K)$, and
$p(K)$ intersects only finitely many tiles $\lambda \cdot F$.
Then $K \cap \Lambda_0$ is finite, so Λ_0 is discrete. $\square$



3) <u>Genus-2 surface.</u>

Take regular 8-gon with angles $\frac{\pi}{4}$.



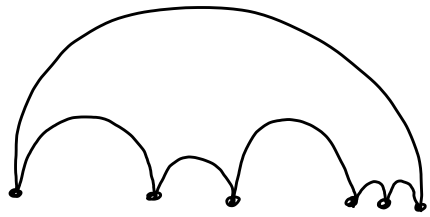
<u>Def</u> A discrete group $\Gamma < SL_2(\mathbb{R})$ is a <u>lattice</u> if $|F| < \infty$ for its fundamental domain.

<u>ex.</u> If F_1 & F_2 are fundamental domains, then $|F_1| = |F_2|$.

<u>Thm.</u> (Siegel) If Γ is a lattice, then the Dirichlet fundamental domain has finitely many sides.

1) D has finitely many vertices on $\mathbb{R} \cup \{\infty\}$.

If $|\partial D \cap (\mathbb{R} \cup \{\infty\})| \geq n$, by convexity, $D \supset$ n-gon with vertices on $\mathbb{R} \cup \{\infty\}$:

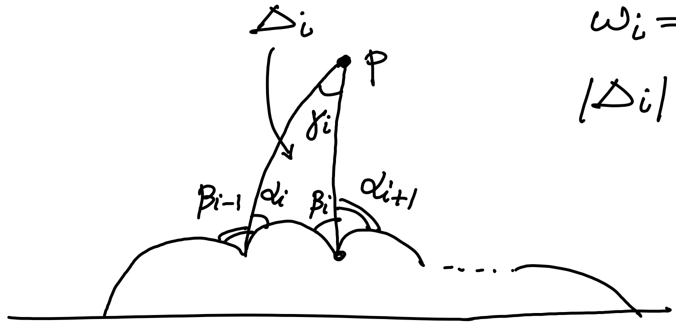


whose area is $\pi(n-2)$.

1) $\Rightarrow$ ∂D has finitely many connected components

2) For all but finitely vertices, angles $\omega_i \geq \frac{3\pi}{4}$.

Consider a piece of D corresponding to a connected component of ∂D:



$\omega_i = \alpha_i + \beta_{i-1}$

$|\Delta_i| = \pi - (\alpha_i + \beta_i + \gamma_i)$

$$\underbrace{\sum_{i=a}^{b} |\Delta_i|}_{\leq |D|} = \sum_{i=a}^{b} (\pi - \alpha_i - \beta_i - \gamma_i) = \pi - \alpha_a - \beta_b + \sum_{i=a+1}^{b-1} (\pi - \omega_i) - \underbrace{\sum_{i=a}^{b} \gamma_i}_{\leq 2\pi}$$

Hence, $\sum_i (\pi - \omega_i) < \infty$ $\Rightarrow$ $\omega_i \geq \frac{3\pi}{4}$ for all but finitely many i.

3) There are only finitely many vertices.

Take a vertex v and $v_1, \dots v_m$ be the other vertices of D in $\Gamma \cdot v$, i.e., $v_i = \gamma_i \cdot v$.

Let $\Gamma_v = \mathrm{Stab}_\Gamma(v)$.

Since Γ is discrete, Γ_v is finite.

All the copies of the tesselation γD adjucent to v are $\gamma \cdot \gamma_i^{-1} D$ with $\gamma \in \Gamma_v$.

Hence, $2\pi = |\Gamma_v| \cdot (\omega + \omega_1 + \dots + \omega_n)$

It follows from 2) that there are only finitely many vertices. $\square$