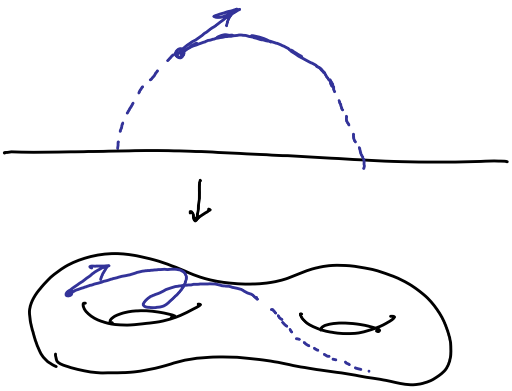
Lecture 3: Geodesic flow.

$\Gamma < SL_2(\mathbb{R})$ – lattice subgroup,

$M = \Gamma \backslash \mathbb{H}$ – hyperbolic subgroup, $|M| < \infty$.

Geodesic flow



$$T^1\mathbb{H} \cong PSL_2(\mathbb{R})$$

$$T^1 M \cong \Gamma \backslash PSL_2(\mathbb{R})$$

$$g_t : T^1 M \to T^1 M$$

$$g_t : \Gamma \backslash PSL_2(\mathbb{R}) \to \Gamma \backslash PSL_2(\mathbb{R})$$

$$x \longmapsto x \cdot \underbrace{\begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}}_{a_t}$$

$$X = T^1 M \simeq \Gamma \backslash PSL_2(\mathbb{R}).$$

Let $\mu = \frac{dx\,dy}{y^2} d\theta$ be the measure on $T^1 M$, where $d\theta$ is the Lebesgue measure on S^1.

For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$, $g : (z, v) \mapsto \left(\frac{az+b}{cz+d}, \frac{v}{(cz+d)^2} \right)$.

It is easy to check that μ is left $SL_2(\mathbb{R})$-inv.,

i.e. defines left inv. measure on $PSL_2(\mathbb{R})$.
Because of invariance, μ also gives a measure on $X = \Gamma \backslash T^1 M$.
Since $|M| < \infty$, $\mu(X) < \infty$.
We normalise μ so that $\mu(X) = 1$.

In fact, μ is also right invariant.
(one can deduce this from uniqueness of inv. measure on $PSL_2(\mathbb{R})$).
In particular, μ is g_t-inv.

<u>Def</u> 1) g_t is <u>ergodic</u> if for every measurable $A \subset X$ which is g_t-inv., we have $\mu(A) = 0$ or 1.

2) g_t is <u>mixing</u> if for every measurable $A, B \subset X$,
$$\mu(A \cap g_t^{-1} B) \to \mu(A)\mu(B) \text{ as } t \to \infty.$$

<u>Thm.</u> The geodesic flow is mixing.

For $g \in SL_2(\mathbb{R})$ and $\varphi \in L^2(X)$, define

$$\pi(g)\varphi(x) = \varphi(x \cdot g).$$

Note that $\pi(g_1 g_2) = \pi(g_1)\pi(g_2)$.

By invariance of μ, $\|\pi(g)\varphi\|_2^2 = \int_X |\varphi(xg)|^2 d\mu(x) = \|\varphi\|_2^2$,

i.e., $\pi(g): L^2(X) \to L^2(X)$ is unitary.

Since $\mu(A \cap g_t^{-1}B) = \langle \chi_A, \pi(a_t)\chi_B \rangle$,

mixing is equivalent to

$$\langle \varphi, \pi(a_t)\psi \rangle \xrightarrow[t\to\infty]{} \left(\int_X \varphi \, d\mu\right)\left(\int_X \overline{\psi} \, d\mu\right).$$

for all $\varphi, \psi \in L^2(X)$.

<u>Lem.</u> The map $SL_2(\mathbb{R}) \to L^2(X) : g \mapsto \pi(g)\varphi$ is continuous.

<u>Hint:</u> $C_c(X) \subset L^2(X)$ is dense.

<u>Proof of Thm.</u>

Without loss of generality, $\int_X \varphi \, d\mu = \int_X \psi \, d\mu = 0$.

Suppose then that for some φ, ψ and $t_n \to \infty$,

$$\langle \varphi, \pi(a_{t_n})\psi \rangle \not\to 0.$$

Weak convergence: $\psi_n \xrightarrow{weak} \psi$ if $\langle \varphi, \psi_n \rangle \to \langle \varphi, \psi \rangle$ for all $\varphi \in L^2(X)$.

Banach-Alaoglu Thm Closed bounded subsets of $L^2(X)$ are compact in weak topology.

Since $\|\pi(a_{t_n})\psi\| = \|\psi\|$, passing to a subsequence, we may assume that $\pi(a_{t_n})\psi \to \tilde{\psi} \in L^2(X)$.

Claim $\tilde{\psi}$ is invariant under $U = \{u_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}\}$.

We use that $a_{t_n}^{-1} u_s a_{t_n} = \begin{pmatrix} e^{t_n/2} & 0 \\ 0 & e^{-t_n/2} \end{pmatrix}^{-1} \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{t_n/2} & 0 \\ 0 & e^{-t_n/2} \end{pmatrix}$

$= \begin{pmatrix} 1 & e^{-t_n}s \\ 0 & 1 \end{pmatrix} = u_{e^{-t_n}s} \to e.$

$$\pi(u_s)\tilde{\psi} = \underset{n\to\infty}{w\text{-}\lim}\ \pi(u_s)\pi(a_{t_n})\psi = \underset{n\to\infty}{w\text{-}\lim}\ \pi(a_{t_n})\pi(u_{e^{-t_n}s})\psi$$

$$\|\pi(a_{t_n})\pi(u_{e^{-t_n}s})\psi - \pi(a_{t_n})\psi\| = \|\pi(u_{e^{-t_n}s})\psi - \psi\| \to 0.$$

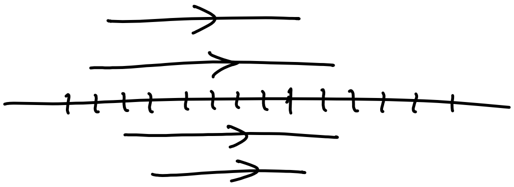
Hence, $\pi(u_s)\tilde{\psi} = \underset{n\to\infty}{w\text{-}\lim}\ \pi(a_{t_n})\psi = \tilde{\psi}$. $\square$

Consider the function $F(g) = \langle \pi(g)\tilde{\psi}, \tilde{\psi} \rangle$, $g \in SL_2(\mathbb{R})$.

Then $F(U \cdot g \cdot U) = F(g)$.

Note that $SL_2(\mathbb{R})/U \simeq \mathbb{R}^2 \setminus \{0\} : gU \mapsto ge_1$,
so F gives a U-inv. function on $\mathbb{R}^2\setminus\{0\}$.

$U \curvearrowright \mathbb{R}^2 : \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x+sy \\ y \end{pmatrix}$



Orbits $\begin{cases} \text{lines } y=c,\ c\neq 0 \\ \text{points } \{(x,0)\}. \end{cases}$

$F = \text{const}$ on $y=c,\ c\neq 0$, and by continuity,
$F = \text{const}$ on $y=0$.

Then $\langle \pi(a_t)\tilde{\varphi}, \tilde{\varphi}\rangle = F(a_t \cdot e_1) = F(e^{t/2}\cdot e_1) = F(e) = \|\tilde{\varphi}\|^2$.

This gives equality in the Cauchy-Schwarz inequality, so $\pi(a_t)\tilde{\varphi} = \lambda\tilde{\varphi}$ and $\lambda = 1$.

Hence, F is B-biinvariant, $B = \left\{\begin{pmatrix} a & x \\ 0 & a^{-1}\end{pmatrix} : a>0\right\}$.

Since $B\cdot e_2 = \{y>0\}$ and $B\cdot(-e_2) = \{y<0\}$,
F is constant on $\{y>0\}$ and $\{y<0\}$, and by continuity, F is constant, i.e.,

$$\langle \pi(g)\tilde{\varphi}, \tilde{\varphi}\rangle = \|\tilde{\varphi}\|^2 \text{ for } g\in SL_2(\mathbb{R}).$$

As above, we deduce that $\pi(g)\tilde{\varphi} = \tilde{\varphi}$ for all g.

This means that for all g and a.e. x, $\tilde{\varphi}(gx) = \tilde{\varphi}(x)$.
Then by Fubini Thm, for a.e. $x \in X$,
$\{g : \tilde{\varphi}(gx) = \tilde{\varphi}(x)\}$ has full measure.
Since G acts transitively, $\tilde{\varphi} = \text{const}$ in $L^2(X)$.
We have $\langle \varphi, \pi(a_{t_n})\psi \rangle \longrightarrow \langle \varphi, \tilde{\psi} \rangle \neq 0$,
but $\underset{\overset{\parallel}{0}}{\langle 1, \pi(a_{t_n})\psi \rangle} \longrightarrow \langle 1, \tilde{\psi} \rangle$.
This is a contradiction. $\square$

Cor. For a.e. $x \in X$, $\{g_t x\}_{t \geq 0}$ is dense in X.

Distribution of orbits.

Mean Ergodic Thm. $\forall \varphi \in L^2(X)$:

$$\left\| \frac{1}{T} \int_0^T \varphi(g_t x)\, dt - \int_X \varphi \, d\mu \right\|_2 \xrightarrow[T \to \infty]{} 0.$$

$\ulcorner$ This follows from mixing. $\lrcorner$

Pointwise Ergodic Thm. $\forall \varphi \in L^2(X)$ $\forall$a.e. $x \in X$:

$$\frac{1}{T}\int_0^T \varphi(g_t \cdot x)\,dt \xrightarrow[T\to\infty]{} \int_X \varphi\, d\mu.$$

Anosow property

Let $u_a = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ and $v_b = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}$.

Stable folliation: $W^s(x) = \{x \cdot u_a : a \in \mathbb{R}\}$.

Unstable folliation: $W^u(x) = \{x \cdot v_b : b \in \mathbb{R}\}$.

Properties:

1) (transversality) $T_x(g_{\mathbb{R}} x) \oplus T_x W^s(x) \oplus T_x W^u(x) = T_x M$.

2) $g_t W^s(x) = W^s(g_t x)$, $g_t W^u(x) = W^u(g_t x)$

$$\ulcorner \quad x u_a a_t = x \cdot a_t \cdot u_{e^{-t} a} \quad \lrcorner$$

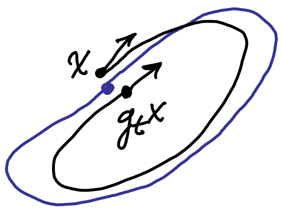
3) (contraction)
for $y \in W^s(x)$, $d(g_t x, g_t y) \leq \text{const} \cdot e^{-t}\, d(x,y)$;
for $y \in W^u(x)$, $d(g_{-t} x, g_{-t} y) \leq \text{const} \cdot e^{-t}\, d(x,y)$.

Let $y = x\cdot u_a$. Then

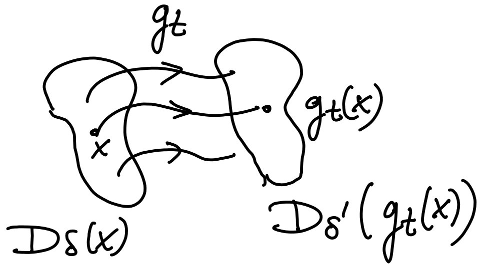
$$d(g_t(x), g_t(y)) = d(x a_t, x u_a \cdot a_t) = d(x a_t, x a_t \cdot u_{ae^{-t}})$$
$$\leq d(e, u_{ae^{-t}}) \leq \text{const}\cdot a e^{-t}.$$

<u>Anosov Closing Lemma.</u>

Suppose that for some $x \in X$ and $t \geq 0$, $d(x, g_t(x)) < \varepsilon$.
Then $\exists$ periodic $x_0 \in X$: $d(x, x_0) \leq \text{const}\cdot\varepsilon$.



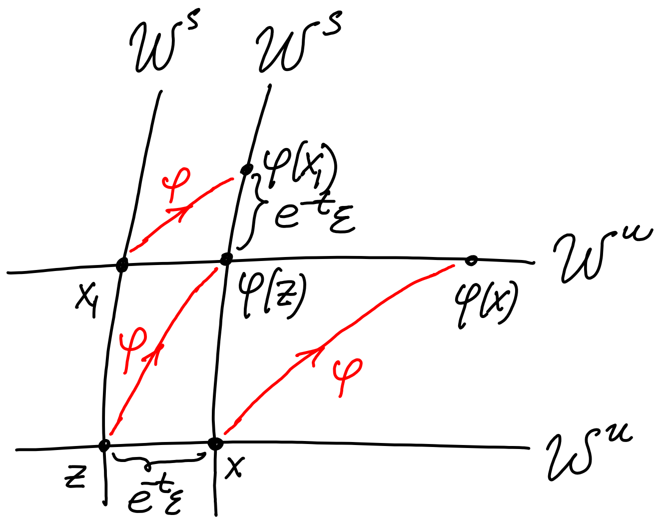
Let $D_\delta(x) = \{x \cdot u_a v_b : |a|, |b| < \delta\}$.
$D_\delta(x)$ is 2-dim. manifold transveral to the flow; $g_t(D_\delta(x)) \subset D_{\delta'}(g_t(x))$.



$\exists t_0 : |t_0| \leq \text{const}\,\varepsilon : g_{t+t_0}(x) \in D_\delta(x)$.

Consider the map:

$$\varphi : D_\delta(x) \to D_{\delta'}(x)$$
$$y \longmapsto g_{t+t_0}(y)$$



By transversality, $\exists\, z \in W^u(x) : \ \varphi(z) \in W^s(x)$.

Then $d(z,x) = d(g_{-t-t_0}(\varphi(z)), g_{-t-t_0}(\varphi(x)))$
$< \text{const} \cdot e^{-t} \cdot \varepsilon$.

Take $x_1 \in W^s(z) \cap W^u(\varphi(z))$.

Then $d(\varphi(x_1), \varphi(z)) \leq d(g_{t+t_0}(x_1), g_{t+t_0}(z))$
$< \text{const} \cdot e^{-t} \cdot \varepsilon$

Hence, $d(x_1, \varphi(x_1)) < \text{const} \cdot e^{-t} \cdot \varepsilon$.

Continuing this process, we construct x_∞ such that $\varphi(x_\infty) = x_\infty$. This gives a periodic orbit. $\square$

<u>Cor.</u> Periodic orbits of geodesic flow are dense.