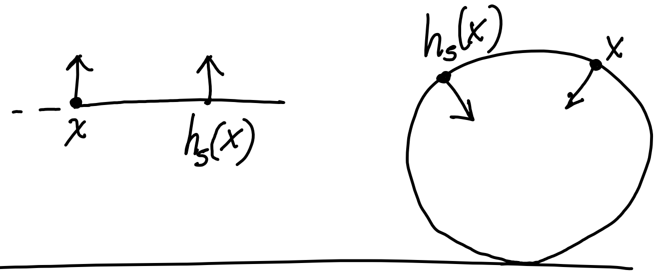
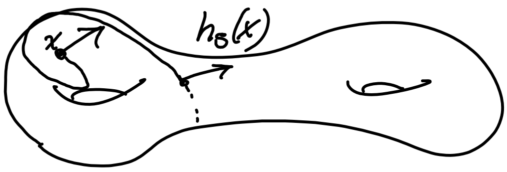
Lecture 4: Horocycle flows.

Γ – lattice subgroup of $SL_2(\mathbb{R})$

$X = \Gamma \backslash SL_2(\mathbb{R})$

Horocycle flows: $h_s : X \to X : x \mapsto x \cdot \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$.





Thm The horocycle flows h_s is mixing.

Lem. (Cartan decomposition)

$$SL_2(\mathbb{R}) = SO(2) \cdot \left\{ a_t = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} : t \geq 0 \right\} \cdot SO(2).$$

<u>Proof of Thm.</u> Suppose for some $\varphi, \psi \in L^2(X)$ and $s_n \to \infty$, $\langle \varphi, \pi(h_{s_n})\psi \rangle \not\to \int_X \varphi \, d\mu \cdot \int_X \bar{\psi} \, d\mu$.

By the Cartan decomposition,

$h_{s_n} = k_n a_{t_n} l_n$ with $k_n, l_n \in SO(2)$ and $t_n \to \infty$.

Then $\langle \varphi, \pi(h_{s_n})\psi \rangle = \langle \pi(k_n)^{-1} \cdot \varphi, \pi(a_{t_n}) \pi(l_n) \psi \rangle$.

Passing to a subsequence, $k_n \to k$ and $l_n \to l$, so that

$$|\langle \pi(k_n)^{-1}\varphi, \pi(a_{t_n})\pi(l_n)\psi \rangle - \langle \pi(k)^{-1}\varphi, \pi(a_{t_n})\pi(l)\psi \rangle|$$

$$\leq |\langle \pi(k_n)^{-1}\varphi - \pi(k)^{-1}\varphi, \pi(a_{t_n})\pi(l_n)\psi \rangle|$$

$$+ |\langle \pi(k)^{-1}\varphi, \pi(a_{t_n})\pi(l_n)\psi - \pi(a_{t_n})\pi(l)\psi \rangle|$$

$$\leq \|\pi(k_n)^{-1}\varphi - \pi(k)^{-1}\varphi\| \cdot \|\psi\| + \|\varphi\| \cdot \|\pi(l_n)\psi - \pi(l)\|$$

$$\to 0.$$

Then $\langle \pi(k)^{-1}\varphi, \pi(a_{t_n})\pi(l)\psi \rangle \not\to \int_X \varphi \, d\mu \cdot \int_X \psi \, d\mu$, which contradicts mixing of g_t. $\square$

<u>Thm</u> (unique ergodicity)

Assume that X is compact.

Then $\forall x\in X\ \ \forall f\in C(X)$:

$$\frac{1}{T}\int_0^T f(h_s(x))\,ds \longrightarrow \int_X f\,d\mu.$$

Let $B=\begin{pmatrix} * & 0\\ * & *\end{pmatrix}\subset Sl_2(\mathbb{R})$ and $B_\delta=\delta$-nbhd of e in B.

Then $G_\delta=h_{[-\delta,\delta]}B_\delta$ is a nbhd of e in G.

Since X is compact, $\exists\delta_0>0$: $G_{\delta_0}\to y\cdot G_{\delta_0}$ is injective for all $y\in X$. (check this).

The measure $\mu|_{yG_\delta}$ is given by $ds\cdot db$ where db is right-invariant measure on B.

Let $Q=h_{[0,\delta_0]}\cdot B_\delta$ and $Q_t=a_tQa_t^{-1}=h_{[0,e^t\delta_0]}\underbrace{(a_tB_\delta a_t^{-1})}_{B_t}$.

Note that $a_t\begin{pmatrix} u & 0\\ w & v\end{pmatrix}a_t^{-1}=\begin{pmatrix} u & 0\\ e^{-t}w & v\end{pmatrix}$, so

B_t is δ-small and $x\cdot Q_t$ is "thickening" of the orbit $x\cdot h_{[0,e^t\delta_0]}$.

By injectivity, $|Q_t|=|Qa_t^{-1}|=|Q|$.

By uniform continuity,

$\forall \varepsilon > 0 : \exists \delta > 0 : \forall y \in X : \forall b \in B_t : \ |f(y \cdot b) - f(y)| < \varepsilon.$

This suggests that $\displaystyle \frac{1}{|yQ_t|} \int_{yQ_t} f \, d\mu \approx \frac{1}{e^{t\delta_0}} \int_0^{e^{t\delta_0}} f(y \cdot a_s) \, ds.$

Indeed,

$$\frac{1}{|yQ_t|} \cdot \int_{yQ_t} f \, d\mu = \frac{1}{|yQ_t|} \cdot \int_0^{e^{t\delta_0}} \int_{B_t} f(y u_s b) \, db \, ds$$

$$(\ast) \qquad = \frac{|B_t|}{e^{t\delta_0} \cdot |B_t|} \cdot \int_0^{e^{t\delta_0}} \big(f(y u_s) + O(\varepsilon) \big) \, ds$$

$$= \frac{1}{e^{t\delta_0}} \int_0^{e^{t\delta_0}} f(y u_s) \, ds + O(\varepsilon).$$

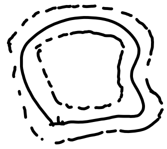
We claim that it follows from mixing (for geodesic flows)

that $\displaystyle \frac{1}{|yQ_t|} \cdot \int_{yQ_t} f \, d\mu = \left\langle f, \frac{\chi_{yQ_t}}{|yQ_t|} \right\rangle \to \int_X f \, d\mu.$

Without loss of generality, $f \geq 0$.

Pick compact Q^- and open Q^+ such that

$Q^- \subset Q \subset Q^+$



$|Q^+ - Q^-| < \varepsilon.$

$\exists \rho>0$: $G_\rho \cdot Q^- \subset Q$ and $G_\rho \cdot Q \subset Q^+$.

By compactness, $X = z_1 G_\rho \cup \ldots \cup z_\ell G_\rho$.

Assuming that $x a_t \in z_i G_\rho$, we obtain:

$$\left\langle f, \frac{\chi_{xQ_t}}{|Q_t|} \right\rangle = \left\langle f, \frac{\chi_{x a_t Q a_t^{-1}}}{|Q_t|} \right\rangle \leq \left\langle f, \frac{\chi_{z_i G_\rho Q a_t^{-1}}}{|Q_t|} \right\rangle$$

$$\leq \left\langle f, \frac{\pi(a_t)\chi_{z_i Q^+}}{|Q_t|} \right\rangle \xrightarrow[t\to\infty]{} \left(\int_X f\right) \cdot \frac{|Q^+|}{|Q_t|}.$$

This proves that

$$\overline{\lim_{t\to\infty}} \left\langle f, \frac{\chi_{xQ_t}}{|Q_t|} \right\rangle \leq \int_X f \cdot (1+\varepsilon)$$

for all $\varepsilon > 0$.

The lower bound is proved similarly.

Hence, the Thm follows from $(*)$. $\square$

Cor. For all $x \in X$, the orbit $\{h_s(x)\}_{s>0}$ is dense.