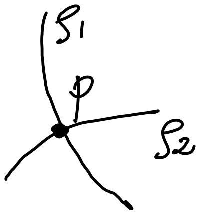
Lecture 5.

Laplace operator and its spectral decomposition.



Let $f \in C^2(\mathbb{H})$.

For $p \in \mathbb{H}$, take orthogonal geodesics γ_1, γ_2 with $\gamma_1(0) = \gamma_2(0) = p$.

The Laplace operator:

$$\Delta f(p) = \frac{d^2 f(\gamma_1(t))}{dt^2}\bigg|_{t=0} + \frac{d^2 f(\gamma_2(t))}{dt^2}\bigg|_{t=0}.$$

Explicitly, the Laplace operator is given by

$$\Delta = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right).$$

We note that Δ is the unique (up to scalar multiple) 2nd order differential operator that commutes with $SL_2(\mathbb{R})$-action.

In particular, for a hyperbolic surface $M = \Gamma \backslash \mathbb{H}$, we have $\Delta : C^\infty(M) \to C^\infty(M)$.

Let $M = \Gamma \backslash \mathbb{H}$ be a compact hyperbolic surface.

Thm (Spectral decomposition)
$L^2(M)$ has an orthonormal basis $\varphi_0, \dots \varphi_n, \dots$ of C^∞-functions with eigenvalues $0 = \lambda_0 < \lambda_1 \leq \dots \leq \lambda_n \leq \dots$ with $\lambda_n \to \infty$.

Prop. 1) Δ is symmetric: $\langle \Delta f_1, f_2 \rangle = \langle f_1, \Delta f_2 \rangle$, $f_1, f_2 \in C^\infty(M)$.
2) Δ is positive: $\langle \Delta f, f \rangle \geq 0$; "$=$" $\Leftrightarrow f = \text{const}$.
$f \in C^\infty(M)$

Consider the differential form:
$$\omega = f_1 \left(\frac{\partial \overline{f_2}}{\partial x} dy - \frac{\partial \overline{f_2}}{\partial y} dx \right) - \overline{f_2} \left(\frac{\partial f_1}{\partial x} dy - \frac{\partial f_1}{\partial y} dx \right).$$

Note that:
1) ω is Γ-invariant (check using the Cauchy-Riemann equations for $z \mapsto \gamma z$)
2) $d\omega = -(f_1 \Delta^e \overline{f_2})\, dx \wedge dy + (\Delta^e f_1 \cdot \overline{f_2})\, dx \wedge dy$,
where $\Delta^e = -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}$ is the Euclidean Laplacian.

By Stoke's Thm, $\int_M d\omega = 0$, and

$$\int_M \Delta^{e} f_1 \cdot \overline{f_2}\, dx\,dy = \int_M f_1 \cdot \Delta^{e} \overline{f_2}\, dx\,dy.$$

$$\quad \| \qquad\qquad\qquad\qquad \|$$

$$\langle \Delta f_1, f_2 \rangle \qquad\qquad \langle f_1, \Delta f_2 \rangle$$

To prove (2), we apply Stoke's Thm to:

$$\omega = \overline{f}\left(\frac{\partial f}{\partial x} dy - \frac{\partial f}{\partial y} dx\right).$$

This form is Γ-inv., and

$$d\omega = \left(\left|\frac{\partial f}{\partial x}\right|^2 + \left|\frac{\partial f}{\partial y}\right|^2 - \overline{f}\, \Delta^{e} f \right) dx \wedge dy.$$

Then $$\int_M \Delta^{e} f \cdot \overline{f}\, dx\,dy = \int_M \left(\left|\frac{\partial f}{\partial x}\right|^2 + \left|\frac{\partial f}{\partial y}\right|^2 \right) dx\,dy \geq 0.$$

$$\|$$

$$\langle \Delta f, f \rangle$$

Hence, $\langle \Delta f, f \rangle \geq 0$ with "$=$" $\Leftrightarrow$ $f = \text{const}$.

The Spectral Thm is proved by studing:

<u>Heat equation:</u> For $f \in C^\infty(M)$, find $u \in C^\infty(M \times \mathbb{R}^+)$,

$$\begin{cases} \Delta u + \dfrac{\partial u}{\partial t} = 0 \\ u|_{t=0} = f. \end{cases}$$

$u(z,t) =$ temperature at time at z,
when the initial temperature is f.

Def. A C^∞-function $p: M\times M\times(0,\infty)\to\mathbb{R}$ is called a heat kernel if:

1) $\Delta p_t(\cdot,w)+\frac{\partial p_t}{\partial t}=0$,
2) $p_t(z,w)=p_t(w,z)$,
3) $\lim\limits_{t\to 0^+}\int_M p_t(z,w)f(w)\,dm(w)=f(z)$.

Note that $u(z,t)=\int_M p_t(z,w)f(w)\,dm(w)$ gives a solution for the heat equation.

Prop. A solution of the heat equation is unique.

If u_1,u_2 are solutions, then $v=u_1-u_2$ is a solution with $f=0$. We note that

$$\frac{d}{dt}\left(\int_M v^2\right)=2\langle v,\frac{\partial v}{\partial t}\rangle=-2\langle v,\Delta v\rangle\leq 0.$$

Since $v|_{t=0}=0$, $\int_M v^2\leq 0$ for $t\geq 0\Rightarrow v=0$.

Cor. 1) The heat kernel is unique.

2) $\int_M p_t(z,w)\,d\mu(w) = 1.$

Proof of Spectral decomposition:

(assuming existance of heat kernel)

We consider a family of operators:

$$\mathcal{P}_t : L^2(M) \longrightarrow L^2(M)$$
$$f \longmapsto \int_M p_t(z,w) f(w)\,dm(w)$$

Properties: 0) $\mathcal{P}_t f$ is a solution of heat equation

1) $\mathcal{P}_{t_1}\mathcal{P}_{t_2} = \mathcal{P}_{t_1+t_2}$

(this follows from uniqueness of solutions of heat equations)

2) $\langle \mathcal{P}_t f, f\rangle = \langle \mathcal{P}_{t/2} f, \mathcal{P}_{t/2} f\rangle \geq 0.$

By the Hilbert-Schmidt Thm, $\mathcal{P}_t$ is diagonalisable, i.e., $L^2(M)$ has an orthonormal basis of eigenfunctions of $\mathcal{P}_t$.

Let $\varphi_0, \varphi_1, \dots \varphi_n, \dots$ be the eigenbasis for $\mathcal{P}_1$ with eigenvalues $\eta_0 \geq \dots \geq \eta_n \geq \dots \geq 0$, $\eta_n \to 0$.

We shall show that $\{\varphi_i\}$ is eigenbasis for Δ.

If $P_{1/k}\varphi = \eta\varphi$, then $P_1\varphi = \eta^k\varphi$.

This implies that the eigenspaces of $P_{1/k}$ and P_1 coinside, and $P_{\ell/k}\varphi_i = \eta_i^{\frac{\ell}{k}}\varphi_i$.

Then by continuity, $P_t\varphi_i = \eta_i^t\varphi_i$ for all $t>0$.

By the properties of heat kernel,

$$P_t\varphi_i = \eta_i^t\varphi_i \underset{t\to\infty}{\longrightarrow} \varphi_i \Rightarrow \eta_i > 0.$$

In particular, $\varphi_i = \eta_i^{-1}P_t\varphi_i \in C^\infty(M)$.

We note that $\varphi = \text{const}$ is an eigenfunction of P_1 with eigenvalue 1. Let φ be $\neq \text{const}$ eigenfunction of P_1 with eigenvalue η. Then

$$\frac{d}{dt}\langle P_t\varphi, P_t\varphi\rangle = 2\left\langle \frac{d}{dt}P_t\varphi, P_t\varphi\right\rangle = -2\langle \Delta P_t\varphi, P_t\varphi\rangle$$
$$= -2\eta^{2t}\langle\Delta\varphi,\varphi\rangle < 0.$$

Hence, $\|P_t\varphi\| = \eta^t\|\varphi\|$ decays, so that $\eta<1$.

Finally, we claim that $\Delta\varphi_i = \lambda_i\varphi_i$ with $\lambda_i = -\log\eta_i$.

Indeed, $0 = \Delta P_t\varphi_i + \frac{\partial}{\partial t}P_t\varphi_i = \Delta(e^{-\lambda_i t}\varphi_i) + \frac{\partial}{\partial t}(e^{-\lambda_i t}\varphi_i)$

$$= e^{-\lambda_i t}(\Delta\varphi_i - \lambda_i\varphi_i). \quad \square$$

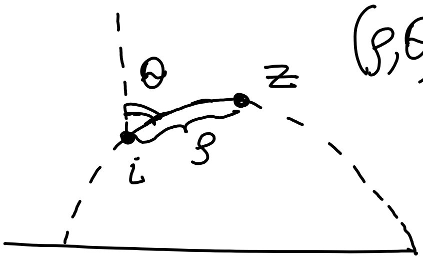
Heat kernel on $\mathbb{H}$.

We are looking for $p_t : \mathbb{H}\times\mathbb{H} \to \mathbb{R}$ satisfying the properties of heat kernel. We expect:

1) $p_t(z,w)$ depends only on $d(z,w)$, namely,
$$p_t(z,w) = P_t(d(z,w)) \text{ for } P_t : [0,\infty) \to \mathbb{R}.$$

2) $P_t(\rho) \to 0$ rapidly as $\rho \to \infty$.

Then it is convenient to use polar coordinates:

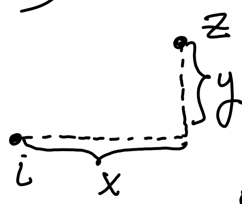


(ρ, θ)

$$\Delta = -\frac{\partial^2}{\partial \rho^2} - \coth(\rho)\cdot\frac{\partial}{\partial \rho} + (\dots)$$

Then $$-\frac{\partial^2 P_t}{\partial \rho^2} - \coth(\rho)\frac{\partial P_t}{\partial \rho} + \frac{\partial P_t}{\partial t} = 0.$$

However, Δ is simpler in horospherical coordinates:



(r, x)

$$\Delta = -\frac{\partial^2}{\partial r^2} + \frac{\partial}{\partial r} + (\dots)$$

$r = \log y$

For a rapidly decaying $f: \mathbb{H} \to \mathbb{R}$, we consider "horospherical transform":

$$(Hf)(z) = \int_{\mathbb{R}} f(z+s)\,ds.$$

Since Hf depends only on r,

$$\begin{tikzcd}
C^\infty(\mathbb{H}) \arrow[r, "H"] \arrow[d, "\Delta"] & C^\infty(\mathbb{H}) \arrow[r, "\times e^{-r/2}"] \arrow[d, "-\frac{\partial^2}{\partial r^2} + \frac{\partial}{\partial r}"] & C^\infty(\mathbb{H}) \arrow[d, "-\frac{\partial^2}{\partial r^2} + \frac{1}{4}"] \\
C^\infty(\mathbb{H}) \arrow[r, "H"] & C^\infty(\mathbb{H}) \arrow[r, "\times e^{-r/2}"] & C^\infty(\mathbb{H})
\end{tikzcd}$$

First, we compute:

$$Q_t(r) = e^{-r/2}\, H[P_t(i, z)],$$

which satisfies the equation:

$$-\frac{\partial^2 Q_t}{\partial r^2} + \frac{1}{4} \cdot Q_t + \frac{\partial Q_t}{\partial t} = 0$$

Since it is similar to the heat equation on $\mathbb{R}$, we look for solutions of the form

$$Q_t(r) = \alpha(t) \cdot e^{-\beta r^2/t},$$

and find $\boxed{Q_t(r) = \text{const} \cdot t^{-1/2} \cdot e^{-t/4} \cdot e^{-r^2/4t}.}$ $(*)$

On the other hand,

$$Q_t(r) = e^{-r/2} \cdot \int_{\mathbb{R}} P_t(d(i, s+ie^r))\, ds$$

Using that $\cosh d(u,v) = 1 + \frac{|u-v|^2}{2\, Im(u) \cdot Im(v)}$,

$$\cosh d(i, s+ie^r) = 1 + \frac{s^2 + (e^r-1)^2}{2e^r} = \cosh r + \tfrac{1}{2} s^2 \cdot e^{-r}.$$

We make change of variables,

$$\rho = d(i, s+ie^r) = \cosh^{-1}\left(\cosh r + \tfrac{1}{2} s^2 \cdot e^{-r}\right).$$

Then $s = \sqrt{2e^r(\cosh\rho - \cosh r)}$ $\quad \sinh\rho\, d\rho = s \cdot e^{-r} dr.$

,

$$\boxed{Q_t(r) = \sqrt{2} \cdot \int_r^\infty \frac{P_t(\rho) \sinh\rho\, d\rho}{\sqrt{\cosh\rho - \cosh r}}} \qquad (**)$$

Def. Abel transform: for rapidly decaying $p: [1,\infty) \to \mathbb{R}$,

$$\mathcal{A}[p](x) = \int_x^\infty \frac{p(y)\, dy}{\sqrt{y-x}} \underset{\uparrow \atop \xi = \sqrt{y-x}}{=} 2 \cdot \int_0^\infty p(x+\xi^2)\, d\xi.$$

Lem. $$a^{-1}[q](x) = -\frac{1}{\pi}\int_x^\infty \frac{q'(y)\,dy}{\sqrt{y-x}}$$

$$p(x) = -\int_0^\infty \left(p(x+\xi^2)\right)'_\xi \,d\xi = -2\int_0^\infty p'(x+\xi^2)\underbrace{\xi\, d\xi}_{\substack{\text{density in}\\ \text{polar coordinates}\\ \text{in } \mathbb{R}^2}}$$

$$= -\frac{4}{\pi}\int_0^\infty\int_0^\infty p'(x+u^2+v^2)\,du\,dv$$

$$= -\frac{2}{\pi}\int_0^\infty a[p'](x+v^2)\,dv = -\frac{2}{\pi}\int_0^\infty a[p]'(x+v^2)\,dv$$

$$\underset{y=x+v^2}{\overset{}{=}} -\frac{1}{\pi}\int_x^\infty \frac{a[p]'(y)}{\sqrt{y-x}}\,dx.$$

By (*) and (**),

$$P_t(\rho) = \text{const}\cdot\int_\rho^\infty \frac{Q_t'(r)}{\sqrt{\cosh r - \cosh\rho}}\,dr = \text{const}\cdot t^{-3/2}\cdot e^{-t/4}\int_\rho^\infty \frac{r\cdot e^{-r^2/4t}}{\sqrt{\cosh r - \cosh\rho}}\,dr,$$

and $p_t(z,w) = P_t(d(z,w))$.

We choose the constant so that

$$\int_{\mathbb{H}} p_t(z,w)\,dm(z) = 1. \quad (***)$$

Lem. $p_t(z,w) \leq \text{const} \cdot t^{-1} \cdot e^{-d(z,w)^2/8t}$. $(****)$

Thm. $$p_t(z,w) = \frac{\sqrt{2}}{(4\pi t)^{3/2}} \cdot e^{-t/4} \cdot \int_{d(z,w)}^{\infty} \frac{r \cdot e^{-r^2/4t}\, dr}{\sqrt{\cosh r - \cosh d(z,w)}}$$
is the heat kernel on $\mathbb{H}$.

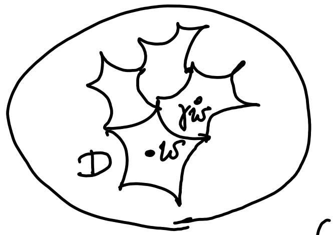
By construction, $p_t(\cdot, w)$ satisfies the heat equation, and clearly, $p_t(z,w) = p_t(w,z)$.
The property: $\int_{\mathbb{H}} p_t(z,w) f(w)\, dm(w) \xrightarrow[t\to 0]{} f(z)$
is deduced from $(***)$ and $(****)$.

Heat kernel on hyperbolic surfaces.

Let $M = \Gamma \backslash \mathbb{H}$ be a compact hyperbolic surface.

We set $\overline{p_t}(z,w) = \sum_{\gamma \in \Gamma} p_t(z, \gamma w)$.

Lem. $\#\{\gamma \in \Gamma : d(z,\gamma w) < T\} \leq c(z,w) \cdot e^T$.



$B_{T+\mathrm{diam}(D)}(w)$

Since $d(z,\gamma w) \geq d(w,\gamma w) - d(z,w)$, we may assume that $z=w$. Let D be the compact Dirichlet fundamental domain at w. Then

$$\#\{\gamma\in\Gamma : d(w,\gamma w)<T\} \leq \text{const}\cdot\Big|\bigcup_{\gamma\in\Gamma:\, d(w,\gamma w)<T}\gamma D\Big|$$

$$\leq \text{const}\,\big|B_{T+\mathrm{diam}(D)}(w)\big|$$

$$\leq \text{const}\cdot e^{T}.$$

By Lemma and $(****)$,

$$\overline{P_t}(z,w) \leq \sum_{n=0}^{\infty} \#\{\gamma\in\Gamma : n\leq d(z,\gamma w)<n+1\}\cdot \text{const}\cdot t^{-1}\cdot e^{-n^2/8t}$$

$$\leq \text{const}\cdot\sum_{n=0}^{\infty} t^{-1}\cdot e^{n}\cdot e^{-n^2/8t} < \infty.$$

Similarly, the sum of derivatives of P_t also converges. Hence, $\overline{P_t}$ satisfies the heat equation.

For $\gamma_1,\gamma_2\in\Gamma$,

$$\overline{P_t}(\gamma_1 z,\gamma_2 w) = \sum_{\gamma\in\Gamma} P_t(\gamma_1 z,\gamma\gamma_2 w) = \sum_{\gamma\in\Gamma} P_t(z,\gamma_1^{-1}\gamma\gamma_2 w)$$

$$= \sum_{\gamma\in\Gamma} P_t(z,\gamma w) = \overline{P_t}(z,w).$$

Hence, $\overline{p_t}$ defines a function on $M \times M$.

Finally, for Γ-inv. $f: \mathbb{H} \to \mathbb{R}$,

$$\int_M \overline{p_t}(z,w) f(w)\, d\mu(w) = \int_D \sum_{\gamma \in \Gamma} p_t(z, \gamma w) f(w)\, dm(w)$$

$$= \int_{\mathbb{H}} p_t(z,w) f(w)\, dm(w) \xrightarrow[t \to 0]{} f(z) \qquad \blacksquare$$