# Lecture 6

## Trace formula and Weyl law.

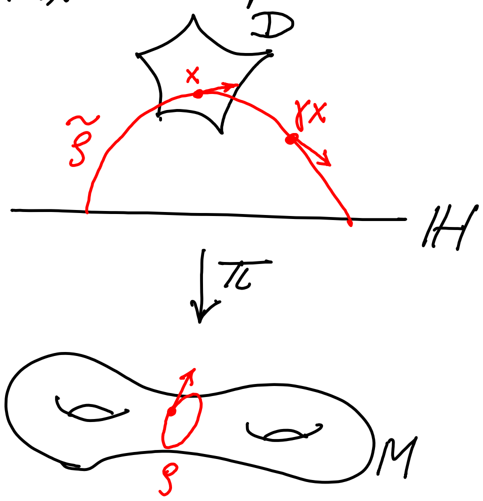
$M = \Gamma\backslash\mathbb{H}$ – compact hyperbolic surface.

$\mathcal{G}(M) = \{$ periodic geodesics in $T^1 M \}$

$$\boxed{\mathrm{Spec}(\Delta) \overset{?}{\longleftrightarrow} \mathcal{G}(M)}$$

Prop. $\#\{\rho \in \mathcal{G}(M) : \underbrace{|\rho|}_{\text{length}} < T\} \leq \mathrm{const} \cdot e^T.$

Fix a compact fundamental domain $D \subset \mathbb{H}$ for $M$.




Given $\rho \in \mathcal{G}(M)$ with $|\rho| < T$, we take a geodesic line $\tilde{\rho} \subset \mathbb{H}$ which is a "lift" of $\rho$, (namely, $\pi(\tilde{\rho}) = \rho$ where $\pi : \mathbb{H} \to M$ is the factor map) and $\tilde{\rho} \cap D \neq \emptyset$.

Then $\exists\, x \in \tilde{\rho} \cap D$, $\gamma \in \Gamma$: as in the picture with $d(x, \gamma x) < T$.

Then $\gamma(\tilde{\rho}) = \tilde{\rho}$. $(*)$

We note that given $\gamma$, $(*)$ determines $\tilde{\rho}$ uniquely.

Indeed, $(*) \Rightarrow \gamma$ fixes the end-points of $\tilde{\rho}$.

This gives a quadratic equation $\gamma \cdot z = z$ which has at most 2 solutions.

Hence, the end-points (and $\tilde{\rho}$) are unique.

Hence, the above correspondence $\rho \longmapsto \gamma$ is 1-to-1, and

$$\#\{\rho \in \mathcal{G}(M) : |\rho| < T\} \leq \#\{\gamma \in \Gamma : d(x, \gamma x) < T \text{ for some } x \in \mathcal{D}\}$$

$$\leq \#\{\gamma \in \Gamma : d(z, \gamma z) < T + \operatorname{diam}(\mathcal{D})\}$$

$$\leq \text{const} \cdot e^T,$$

for a fixed $z \in \mathcal{D}$. $\square$

<u>Def</u> 1) $g \in SL_2(\mathbb{R})$ is <u>hyperbolic</u> if it can be conjugated to a diagonal matrix.

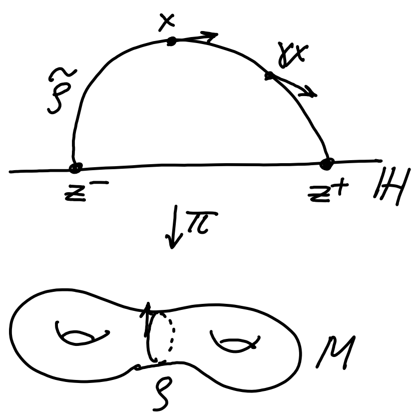
2) $\gamma \in \Gamma$ is <u>primitive</u> if $\gamma \neq \delta^k$ for some $k \geq 2$, $\delta \in \Gamma$.

We note every hyperbolic $g$ has 2 fixed points in $\mathbb{R} \cup \{\infty\}$ and fixes the unique geodesic

**Thm.** There is a 1-to-1 correspondence:

$$\mathcal{G}(M) \longleftrightarrow \left\{ \begin{array}{l} \text{congugacy classes in } \Gamma \\ \text{of primitive hyperbolic elements} \end{array} \right\}$$

$\ulcorner$




As in the previous argument, given $\rho \in \mathcal{G}(M)$, we take a geodesic line $\tilde{\rho} \subset \mathbb{H}$ such that $\pi(\tilde{\rho}) = \rho$, and $\gamma \in \Gamma \setminus \{e\}$ such that $\gamma \cdot \tilde{\rho} = \tilde{\rho}$.

Let $A_{\tilde{\rho}} = \mathrm{Stab}_{PSL_2(\mathbb{R})}(\tilde{\rho})$. We note that $A_{\{x=0\}} = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$ and $A_{\tilde{\rho}}$ is conjugate to $\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$.

In particular, $A_{\tilde{\rho}} \simeq \mathbb{R}$. Since $A_{\tilde{\rho}} \cap \Gamma$ is discrete, it must be cyclic and $A_{\tilde{\rho}} \cap \Gamma = \langle \gamma_0 \rangle$ for $\gamma_0 \in \Gamma \setminus \{e\}$.

We claim that $\gamma_0$ is primitive. Indeed, if $\gamma_0 = \delta^k$ for some $\delta \in \Gamma$ and $k \geq 2$, then $\delta$ is also hyperbolic and $\delta \cdot \tilde{\rho} = \tilde{\rho}$, i.e. $\delta \in A_{\tilde{\rho}} \cap \Gamma$, but this is impossible because $\gamma_0$ is a generator $A_{\tilde{\rho}} \cap \Gamma \simeq \mathbb{Z}$.

Now we constructed a map $\rho \longmapsto \gamma_0$.

Conversely, given a primitive hyperbolic $\gamma_0 \in \Gamma$, it has 2 fixed points $z^-, z^+ \in \mathbb{R} \cup \{\infty\}$ (attracting & repelling), which determines a unique directed geodesic line $\tilde{\rho}$ and a unique closed geodesic in $T^1(M)$.

The correspondence $\tilde{\rho} \longleftrightarrow \gamma_0$ is 1-to-1, but it might happen that

$$\pi(\tilde{\rho}_1) = \pi(\tilde{\rho}_2) \Leftrightarrow \exists \gamma \in \Gamma : \gamma \cdot \tilde{\rho}_1 = \tilde{\rho}_2 .$$

Then $A_{\tilde{\rho}_2} = \gamma A_{\tilde{\rho}_1} \gamma^{-1}$ and $\underset{\langle \gamma_2 \rangle}{\underset{\shortparallel}{A_{\tilde{\rho}_2} \cap \Gamma}} = \gamma \cdot \underbrace{(A_{\tilde{\rho}_1} \cap F)}_{\shortparallel \atop \langle \gamma_1 \rangle} \cdot \gamma^{-1}$.

Hence, $\gamma_2^{\pm 1} = \gamma \cdot \gamma_1 \cdot \gamma^{-1}$.

In fact, $\gamma_2 = \gamma \cdot \gamma_1 \cdot \gamma^{-1}$ if we take directions of $\tilde{\rho}_1$ and $\tilde{\rho}_2$ into account.

## Length Trace Formula.

Let $K : [0,\infty) \to \mathbb{R}$ such that $|K(\rho)| \leq \text{const} \cdot e^{-(1+\varepsilon)\rho}$, $\varepsilon > 0$, and $k(z,w) = K(d(z,w))$, $z, w \in \mathbb{H}$.

For a compact hyperbolic surface $M = \Gamma \backslash \mathbb{H}$,
we define : $\bar{k}(z,w) = \sum_{\gamma \in \Gamma} k(z, \gamma w)$.

Since $\#\{\gamma \in \Gamma : d(z, \gamma w) < T\} \leqslant \text{const} \cdot e^T$,
the sum converges and defines a kernel on $M$.

We consider the integral operator:

$$\mathcal{K}_M : f \longmapsto \int_M \bar{k}(z,w) f(w)\, d\mu(w).$$

<u>Thm</u> (length trace formula)

Assume that $\Gamma \setminus \{e\}$ consists of hyperbolic elements.

Then:

$$\boxed{\begin{aligned} \mathrm{Tr}(K_M) &= \int_M \bar{k}(z,z)\, d\mu(z) = \\ &= |M| \cdot K(0) + \sum_{n \geq 1} \sum_{\sigma \in \mathcal{L}(M)} \frac{|\sigma|}{\sqrt{\cosh(n|\sigma|) - 1}} \cdot \int_{n|\sigma|}^{\infty} \frac{K(\rho) \sinh \rho \, d\rho}{\sqrt{\cosh \rho - \cosh(n|\sigma|)}}. \end{aligned}}$$

Let $\Pi$ be a set of representatives of conjugacy classes of primitive elements in $\Gamma$.

Then $\forall \gamma \in \Gamma \setminus \{e\}$: $\gamma = r^{-1} s^n r$ for $r \in \Gamma$, $s \in \Pi$, $n \in \mathbb{N}$, where $s$ and $n$ are uniquely determined.

We note that $r_1^{-1} s r_1 = r_2^{-1} s r_2 \Leftrightarrow r_1 r_2^{-1} \in Z_s$, where $Z_s$ = the centraliser of $s$ in $\Gamma$.

Let $R_s$ be a set of representatives for cosets of $Z_s \backslash \Gamma$.

Then $\Gamma \setminus \{e\} = \bigsqcup_{n \geq 1,\, s \in \Pi,\, r \in R_s} \{r^{-1} \cdot s^n \cdot r\}$, and

$$\int_M \bar{k}(z,z)\, d\mu(z) \underset{\substack{\uparrow \\ F = \text{a compact} \\ \text{fundamental domain}}}{=} \int_F \Big(\sum_{\gamma \in \Gamma} k(z, \gamma z)\Big)\, dm(z)$$

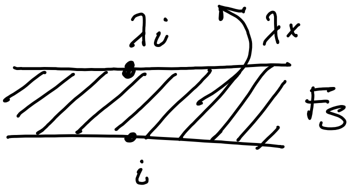
$$= |F| \cdot K(0) + \sum_{n \geq 1,\, s \in \Pi} \Big( \sum_{r \in R_s} \int_F k(z, r^{-1} s^n r z)\, dm(z) \Big).$$

$$\sum_{r \in R_s} \int_F k(z, r^{-1} s^n r z)\, dm(z) = \sum_{r \in R_s} \int_F k(rz, s^n r z)\, dm(z)$$

$$= \int_{R_s F} k(z, s^n z)\, dm(z).$$

We note that $R_s F$ is a fundamental domain for $Z_s$, and $\int_{R_s F} k(z, s^n z)\, dm(z) = \int_{F_s} k(z, s^n z)\, dm(z)$ for <u>any</u> fundamental domain $F_s$ of $Z_s$.

Since $m$ is $SL_2(\mathbb{R})$-inv., we may replace $S$ by its conjugate and assume that




$$Sz = \lambda z, \ \lambda > 1,$$

$$F_S = \{ z \in \mathbb{H} : 1 \le \mathrm{Im}(z) < \lambda \}.$$

Then
$$\int_{F_S} k(z, \lambda^n z)\, dm(z) = \int_1^{\lambda} \int_{\mathbb{R}} K(d(z, \lambda^n z)) \frac{dx\,dy}{y^2}$$

$$= \int_1^{\lambda} \int_{\mathbb{R}} K\left(d\left(i, \frac{(\lambda^n - 1)x}{y} + i\lambda^n\right)\right) \frac{dx\,dy}{y^2}$$

$z \mapsto \frac{z-x}{y}$, $\lambda^n z \mapsto \frac{\lambda^n z - x}{y}$ — isometry

$$= \int_1^{\lambda} \int_{\mathbb{R}} K(d(i, a + i\lambda^n)) \frac{dy}{y}\, da$$

$a = \frac{(\lambda^n - 1)x}{y}$

$$= \frac{\log \lambda}{\lambda^n - 1} \cdot \int_{\mathbb{R}} K(d(i, a + i\lambda^n))\, da.$$

This is "horospherical transform", which we have computed previously (see $(**)$):

$$\int_{\mathbb{R}} K(d(i, a+i\lambda^n))\,da = \sqrt{2}\cdot\lambda^{n/2}\int_{\log(\lambda^n)} \frac{K(\rho)\sinh\rho\, d\rho}{\sqrt{\cosh\rho - \cosh(\log(\lambda^n))}}.$$

We note that $\lambda = e^{|\sigma|}$ where $|\sigma|$ is the length of the corresponding periodic geodesic. Hence,

$$\int_{F_S} k(z, S^n z)\,dm(z) = \frac{2^{-1/2}\cdot|\sigma|}{|\sinh(n|\sigma|/2)|}\cdot\int_{n|\sigma|}^{\infty}\frac{K(\rho)\sinh\rho\, d\rho}{\sqrt{\cosh\rho - \cosh(n|\sigma|)}}.$$

$\lrcorner$

Thm. (spectral trace formula)
Let $P_t$ be the integral operator defined by the heat kernel. Then

$$\mathrm{Tr}(P_t) = \sum_{\lambda\in \mathrm{Spec}(\Delta)} e^{-\lambda t}.$$

$\ulcorner$ Let $\{\varphi_i\}$ be the orthonormal eigenbasis for $\Delta$. Then $\overline{p_t}(z,\cdot) = \sum_i c_i(t,z)\cdot\varphi_i$, where $c_i(t,z) = \langle \overline{p_t}(z,\cdot), \varphi_i\rangle = P_t\varphi_i$ is the solution of the heat equation with the initial condition $f=\varphi_i$, so that $c_i(t,z)=e^{-\lambda_i t}\varphi_i$, and $\overline{p}_t(z,w)=\sum_{i\geq 0} e^{-\lambda_i t}\varphi_i(z)\varphi_i(w)$.

One can show that the convergence is uniform.

Hence, 
$$\mathrm{Tr}(P_t)=\int_M \overline{p}_t(z,z)\,d\mu(z)=\sum_i e^{-\lambda_i t}\int_M \varphi_i^2\,d\mu$$
$$=\sum_i e^{-\lambda_i t}.$$

<u>Cor</u> (heat trace formula)

$$\sum_{\lambda\in \mathrm{Spec}(\Delta)} e^{-\lambda t}=|M|\cdot(4\pi t)^{-3/2}\cdot e^{-t/4}\int_0^\infty \frac{r\cdot e^{-r^2/4t}}{\sinh(r/2)}\,dr$$
$$+\frac{1}{2}(4\pi t)^{-1/2}e^{-t/4}\cdot\sum_{n\geq 1}\sum_{\sigma\in\mathcal{P}(M)}\frac{|\sigma|}{\sinh\left(\frac{n|\sigma|}{2}\right)}\cdot e^{-\frac{n^2|\sigma|^2}{4t}}.$$

Combine the previous theorems and the formula for the inverse Abel transform.

<u>Cor</u> (Weyl law)

$$N(T) = \#\{\lambda \in \mathrm{Spec}(\Delta) : \lambda < T\} \sim \frac{|M|}{4\pi} \cdot T \quad \text{as } T \to \infty.$$

The trace formula gives the asymptotics for the Laplace transform $\int_0^\infty e^{-tx}\, dN(x)$ as $t \to 0^+$.

We obtain:

"1st term" $\sim \frac{|M|}{4\pi} \cdot t^{-1}$ as $t \to 0^+$

"2nd term" $\longrightarrow 0$ as $t \to 0^+$.

Now the statement follows from the Tauberian Thm.