Lecture 7.

Microlocal lifts and quantum ergodicity.

$M = \Gamma \backslash \mathbb{H}$ – compact hyperbolic surface.

$\Delta : C^\infty(M) \to C^\infty(M)$ – Laplace operator

$\{\varphi_n\}$ – orthonormal basis of eigenfunctions of Δ with eigenvalues $\lambda_i \to \infty$.

Consider the sequence of prob. measures:

$$\mu_n = |\varphi_n|^2 d\mu \quad \text{on } M.$$

μ_n is the "distribution" of the position of the particle with energy λ.

<u>Conj.</u> (Quantum unique ergodicity / Rudnick–Sarnak)

$$\boxed{\mu_n \xrightarrow[n\to\infty]{} \text{Area}}$$

i.e., $\forall f \in C(M)$: $\int_M f\, d\mu_n \to \int_M f\, d\mu$.

$\mu_n \xrightarrow[\text{Microlocal lift}]{} \nu_n$ = prob. measures on $T^1M \simeq \Gamma \backslash SL_2(\mathbb{R})$:

- ν_n project to μ_n,
- limits of ν_n are invariant under the geodesic flow.

Differential operators.

$G = SL_2(\mathbb{R})$

$\mathfrak{g} = \{ X \in M_2(\mathbb{R}) : \exp(tX) \in G \text{ for all } t \in \mathbb{R} \}$,

where $\exp(Y) = \sum_{i=0}^{\infty} \frac{Y^i}{i!}$ is the exponential map.

↑

Lie algebra of G.

ex. Show that $\mathfrak{g} = \{ X : \mathrm{Tr}(X) = 0 \}$.

Def For $X \in \mathfrak{g}$, we define differential operator:

$$D_X : C^\infty(G) \to C^\infty(G) : f \mapsto \frac{d}{dt} f(g \cdot \exp(tX)) \Big|_{t=0}.$$

Properties:

1) $D_{\alpha X + \beta Y} = \alpha D_X + \beta D_Y$,

2) $\pi(g) D_X \pi(g^{-1}) = D_{gXg^{-1}}$,

3) $D_X^* = D_{-X}$,

4) $D_X D_Y - D_Y D_X = D_{[X,Y]}$,

where $[X,Y] = XY - YX \in \mathfrak{g}$.

Proof (4): $D_X D_Y f(g) = \frac{\partial^2}{\partial t_1 \partial t_2} f(g \exp(t_2 X) \exp(t_1 Y))\Big|_{t_1=t_2=0}$.

$$\exp(t_2 X)\exp(t_1 Y) = \exp(t_1 Y)\cdot \exp(t_2 \cdot \underbrace{\exp(t_1 Y)^{-1} X \exp(t_1 Y)}_{X_{t_1}}),$$

where $X_{t_1} = (I - tY + \dots) X (I + tY + \dots) = I + t \cdot [X, Y] + O(t^2)$.

Hence, by the chain rule,

$$D_X D_Y f(g) = \frac{\partial^2}{\partial t_1 \partial t_2} f(g \exp(t_1 Y) \exp(t_2 X_{t_1}))\Big|_{t_1=t_2=0}$$

$$= \frac{d}{dt_1} D_{X_{t_1}} f(g \exp(t_1 Y))\Big|_{t_1=0}$$

$$= D_{[X,Y]} f + D_Y D_X f.$$

We also define D_X for $X \in \mathfrak{g} \otimes \mathbb{C}$ by linearity.

Notation: $H = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix} \leftarrow$ direction of the geodesic flow,

$U^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $U^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \leftarrow$ direction of horocycle flows,

$W = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \leftarrow$ direction of $K = SO(2)$.

$$\Omega = D_H D_H + \tfrac{1}{2} D_{V+} D_{V-} + \tfrac{1}{2} D_{V-} D_{V+}$$

Casimir operator

ex. 1) Ω - commutes with all other D_X,

2) $\Omega\big|_{C^\infty(G)^K} = \Delta$, where we identify $G/K \simeq \mathbb{H}$.

Fourier Expansion.

Let $k_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \in K = SO(2)$.

For $f \in C^\infty(G)$, define

$$f_n(g) = \int_K f(g k_\theta) \cdot e^{-in\theta} \frac{d\theta}{2\pi}.$$

Then $f_n(g k_\sigma) = f_n(g) \cdot e^{in\sigma}$, and

$$\boxed{f(g) = \sum_{n \in \mathbb{Z}} f_n(g)}$$

Let $\mathcal{A}_n = \{ f : f(g k_\theta) = f(g) \cdot e^{in\theta} \}$.

Since D_W is the derivative along k_θ,

$$\mathcal{A}_n = \{ f : D_W f = in f \}.$$

Def. f is <u>K-finite</u> if it lies in a span of finitely many $\mathcal{A}_n$'s.

The linear map $X \mapsto [W, X]$ can be diagonalised: for $E^+ = \frac{1}{2}\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}$ and $E^- = \frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}$,

$$[W, E^{\pm}] = \pm 2i \, E^{\pm}.$$

This implies that

$$D_{E^\pm}(\mathcal{A}_n) \subset \mathcal{A}_{n\pm 2}.$$

Indeed, for $f \in \mathcal{A}_n$,

$$D_W D_{E^+} f = D_{E^+} D_W f + D_{[W, E^+]} f = (n+2) i \cdot D_{E^+} f.$$

$\{E^+, E^-, W\}$ forms a basis of $\mathfrak{g} \otimes \mathbb{C}$, and

$$\Omega = D_{E^+} D_{E^-} - \frac{1}{4} D_W^2 + \frac{i}{2} D_W$$
$$= D_{E^-} D_{E^+} - \frac{1}{4} D_W^2 - \frac{i}{2} D_W.$$

Microlocal lift.

Let $\varphi \in C^\infty(\Gamma \backslash \mathbb{H})$ be an eigenfunction of Δ with eigenvalue $\lambda = \frac{1}{4} + r^2$, $\|\varphi\|_2 = 1$.

We aim to construct a prob. measure on $\Gamma \backslash G$ which projects to $|\varphi|^2 d\mu$ and is assymptotically invariant under the geodesic flow as $\lambda \to \infty$.

First, we construct a "distribution" I_φ.

We define inductively:

$$\varphi_0(g) = \varphi(gK),$$

$$\varphi_{2n+2} = \frac{1}{ir + \frac{1}{2} + n} D_{E^+} \varphi_{2n}, \quad n \geq 0,$$

$$\varphi_{2n-2} = \frac{1}{ir + \frac{1}{2} - n} D_{E^-} \varphi_{2n}, \quad n \leq 0.$$

Since Ω commutes with $D_{E^\pm}$, $\Omega \varphi_{2n} = \lambda \varphi_{2n}$.

Then $\|D_{E^+} \varphi_{2n}\|^2 = \langle D_{E^+}^* D_{E^+} \varphi_{2n}, \varphi_{2n} \rangle = -\langle D_{E^-} D_{E^+} \varphi_{2n}, \varphi_{2n} \rangle$

$$= -\langle (\Omega + \tfrac{1}{4} D_W^2 + \tfrac{i}{2} D_W) \varphi_{2n}, \varphi_{2n} \rangle$$

$$= (-\lambda + \tfrac{1}{4} n^2 + \tfrac{1}{2} n) \|\varphi_{2n}\|^2 = |ir + \tfrac{1}{2} + \tfrac{1}{2} n|^2 \cdot \|\varphi_{2n}\|^2.$$

Hence, $\|\varphi_n\|_2 = 1$.

For K-finite $f \in C^\infty(\Gamma\backslash G)$, we define

$$I_\varphi(f) = \langle f \cdot \underbrace{\sum_{n\in\mathbb{Z}} \varphi_{2n}}_{\varphi_\infty}, \varphi_0 \rangle .$$

Since f is K-finite, $\langle f\varphi_{2n}, \varphi_0 \rangle = 0$ for all but finitely many n.

Clearly, for $f \in \mathcal{A}_0$,

$$\boxed{I_\varphi(f) = \langle f, |\varphi|^2 \rangle = \int_M f \, d\mu_\lambda .}$$

In fact, I_φ is asymptotically a measure:

Lem. 1. Let $\psi = \frac{1}{\sqrt{2N+1}} \cdot \sum_{n=-N}^{N} \varphi_{2n}$ with $N = [\sqrt{r}]$,

$\nu = |\psi|^2 dm$ – prob. measure on $\Gamma\backslash G$.

Then for K-finite $f \in C^\infty(\Gamma\backslash G)$,

$$\boxed{I_\varphi(f) = \int_{\Gamma\backslash G} f \, d\nu + O_f(r^{-1/2}).} \qquad (*)$$

We have $\int_{\Gamma\backslash G} f\, d\nu = \langle f\psi, \psi\rangle = \frac{1}{2N+1} \cdot \sum_{n,m=-N}^{N} \langle f\varphi_{2n}, \varphi_{2m}\rangle.$

Suppose that $f \in \sum_{\ell=-2L}^{2L} \mathcal{A}_\ell$. Then

$$\langle f\varphi_{2n}, \varphi_{2m}\rangle = 0 \quad \text{for } |n-m| > L$$

because the spaces $\mathcal{A}_\ell$ are orthogonal.

For $|n-m| \leq L$,

$$\langle f\varphi_{2n}, \varphi_{2m}\rangle = \frac{1}{ir+\frac{1}{2}+n-1} \langle f D_{E^+}\varphi_{2n-2}, \varphi_{2m}\rangle$$

$$= \frac{1}{ir-\frac{1}{2}+n} \cdot \Big[\langle D_{E^+}(f\varphi_{2n-2}), \varphi_{2m}\rangle - \underbrace{\langle D_{E^-}(f)\varphi_{2n-2}, \varphi_{2m}\rangle}_{\text{Bounded}}\Big]$$

$$= \frac{1}{ir-\frac{1}{2}+n} \cdot \langle f\varphi_{2n-2}, \underset{\substack{\| \\ -D_{E^-}}}{D_{E^+}^{*}}\varphi_{2m}\rangle + O(r^{-1})$$

$$= -\underbrace{\frac{-ir+\frac{1}{2}-m}{ir-\frac{1}{2}+n}}_{1+O\left(\frac{|m-n|}{r}\right)} \langle f\varphi_{2n-2}, \varphi_{2m-2}\rangle + O(r^{-1})$$

$$= \langle f\varphi_{2n-2}, \varphi_{2m-2}\rangle + O(r^{-1}).$$

$$= \langle f\varphi_{2(n-m)}, \varphi_0\rangle + O(\underbrace{Nr^{-1}}_{r^{-1/2}})$$

Hence, $\langle f\psi, \psi\rangle = \frac{1}{2N+1} \cdot \sum_{\substack{m=n=-N \\ |m-n|\le L}}^{N} \left(\langle f\varphi_{2(n-m)}, \varphi_0\rangle + O(r^{-1/2})\right)$

$$= \sum_{\ell=-L}^{L} \underbrace{\frac{2N+1-|\ell|}{2N+1}}_{1+O(N^{-1})} \left(\langle f\varphi_{2\ell}, \varphi_0\rangle + O(r^{-1/2})\right)$$

$$= \underbrace{\left\langle f \cdot \sum_{\ell=-L}^{L} \varphi_{2\ell}, \varphi_0 \right\rangle}_{I_\varphi(f).} + O(r^{-1/2})$$

$\square$

<u>Lem. 2</u> $\exists$ fixed differential operator $\mathcal{L}$:

$\forall K$-finite $f \in C^\infty(\Gamma\backslash G)$: $\boxed{I_\varphi\left((rD_H + \mathcal{L})f\right) = 0.}$ (**)

Recall that $I_\varphi(f) = \langle f \underbrace{\varphi_\infty}_{\substack{\| \\ \sum_{n\in\mathbb{Z}} \varphi_{2n}}}, \varphi_0 \rangle.$

We obtain:

$$\lambda\langle f\varphi_\infty, \varphi_0\rangle = \langle f\varphi_\infty, \Omega\varphi_0\rangle = \langle f\varphi_\infty, D_{E^-}D_{E^+}\varphi_0\rangle$$

$$= \langle \underbrace{(D_{E^-}D_{E^+})^*}_{D_{E^-}D_{E^+}}(f\varphi_\infty), \varphi_0\rangle = \langle D_{E^-}D_{E^+}(f)\cdot\varphi_\infty, \varphi_0\rangle +$$

$$+ \langle D_{E^+}(f) D_{E^-}(\varphi_\infty), \varphi_0 \rangle + \langle D_{E^-}(f) D_{E^+}(\varphi_\infty), \varphi_0 \rangle + \langle f D_{E^-} D_{E^+}(\varphi_\infty), \varphi_0 \rangle,$$

where

$$D_{E^-}(\varphi_\infty) = \sum_{n \in \mathbb{Z}} (ir + \tfrac{1}{2} - n) \underbrace{\varphi_{2n-2}}_{\substack{\cap \\ a_{2n-2}}} = (ir + \tfrac{i}{2} D_W - \tfrac{1}{2}) \varphi_\infty,$$

$$D_{E^+}(\varphi_\infty) = \sum_{n \in \mathbb{Z}} (ir + \tfrac{1}{2} + n) \varphi_{2n+2} = (ir - \tfrac{i}{2} D_W - \tfrac{1}{2}) \varphi_\infty,$$

$$D_{E^-} D_{E^+}(\varphi_\infty) = (\Omega - \tfrac{1}{4} D_W^2 - \tfrac{i}{2} D_W) \varphi_\infty = \lambda \varphi_\infty - (\tfrac{1}{4} D_W^2 + \tfrac{i}{2} D_W) \varphi_\infty.$$

Then λ cancels out, and the equality becomes:

$$ir \Big(\underbrace{\langle D_{E^+}(f) \varphi_\infty, \varphi_0 \rangle + \langle D_{E^-}(f) \varphi_\infty, \varphi_0 \rangle}_{2 i\, I_\varphi(D_H f)} \Big) + \underset{\substack{\uparrow \\ \text{independent of } r}}{(\text{—''—})} = 0$$

We note that (—''—) contains only the diff. operator D_W. Since $D_W(\varphi_0) = 0$,

$$0 = -\langle D_W(f_1 f_2), \varphi_0 \rangle = -\langle D_W(f_1) f_2, \varphi_0 \rangle - \langle f_1 D_W(f_2), \varphi_0 \rangle,$$

and $\langle f_1 D_W(f_2), \varphi_0\rangle = -\langle D_W(f_1) f_2, \varphi_0\rangle$.

Using this identity, we obtain

$$(\text{—}''\text{—}) = \langle \mathcal{L}(f)\varphi_\infty, \varphi_0\rangle = I_\varphi(\mathcal{L}(f)).$$

where $\mathcal{L}$ is an explicit diff. operator.

Combining $(*)$ & $(**)$,

$$\int_{\Gamma\backslash G} D_H(f)\, d\nu = \int_{\Gamma\backslash G} f\, d\nu + O(r^{-1/2}).$$

Hence, limits of ν as $\lambda\to\infty$ are invariant under the geodesic flow.

Quantum Ergodicity.

Thm. For every K-finite $f\in C^\infty(\Gamma\backslash G)$,

$$\underbrace{\frac{1}{\#\{\lambda\in \mathrm{Spec}(\Delta) : \lambda\le L\}}}_{N(L)} \cdot \sum_{\lambda\in\mathrm{Spec}(\Delta):\lambda\le L} \left| \int_{\Gamma\backslash G} f\, d\nu_\lambda - \int_{\Gamma\backslash G} f \right|^2 \xrightarrow[L\to\infty]{} 0,$$

where $d\nu_\lambda(g) = |\psi_\lambda(g)|^2 dg$.

In the proof, we use:

<u>General Weyl Law:</u> For every K-finite f,

$$\sum_{\lambda \in Spec(\Delta):\, \lambda \leq L} \langle f, |\psi_\lambda|^2 \rangle \sim \left(\int_{\Gamma \backslash G} f \right) \cdot \frac{vol(G/\Gamma)}{4\pi} \cdot L .$$

$\ulcorner$ We may assume that $\int_{\Gamma \backslash G} f = 0$.

Let $A_T(f) = \frac{1}{T} \int_0^T \pi(a_t) f \, dt$.

$\uparrow$ geodesic flow

By Lemma 1-2 and the Mean Value Thm:

$$\langle \pi(a_t) f, |\psi_\lambda|^2 \rangle = \langle f, |\psi_\lambda|^2 \rangle + O(t \cdot \lambda^{-1/4}),$$

$$\langle A_T(f), |\psi_\lambda|^2 \rangle = \langle f, |\psi_\lambda|^2 \rangle + O(T \lambda^{-1/4}).$$

Hence,

$$\sum_{L_0 < \lambda \leq L} |\langle f, |\psi_\lambda|^2 \rangle|^2 = \sum_{L_0 < \lambda \leq L} |\langle A_T(f), |\psi_\lambda|^2 \rangle|^2 + O_T(L \cdot L_0^{-1/4})$$

$$\leq \sum_{L_0 < \lambda \leq L} \langle |A_T(f)|^2, |\psi_\lambda|^2 \rangle + O_T(L \cdot L_0^{-1/4}),$$

$\uparrow$ Cauchy-Schwarz inequality

and by the Weyl law,

$$\overline{\lim_{L\to\infty}} \frac{1}{N(L)} \cdot \sum_{\lambda\leq L} |\langle f, |\psi_\lambda|^2\rangle|^2 \leq \|A_T(f)\|_2^2 .$$

Finally, by ergodicity (or mixing) of the geodesic flow, $\|A_T(f)\|_2 \xrightarrow[T\to\infty]{} 0$.