Lecture 8

Hecke operators and recurrence of eigenstates.

For $r \in \mathbb{Q}$, write $r = p^n \cdot \frac{\ell}{s}$ with ℓ, s coprime to p.

Define p-adic norm: $|r|_p = p^{-n}$.

Basic properties: $|r_1 \cdot r_2|_p = |r_1|_p \cdot |r_2|_p$,

$|r_1 + r_2|_p \leq \max\{|r_1|_p, |r_2|_p\}$.

$\mathbb{Q}_p$ = completion of $\mathbb{Q}$ with respect to $|\cdot|_p$.

the field of p-adic numbers

$\cup$

$\mathbb{Z}_p = \{x : |x|_p \leq 1\}$ - compact open subring.

ex. $\mathbb{Z}[\frac{1}{p}] \underset{diag}{\hookrightarrow} \mathbb{R} \times \mathbb{Q}_p$ is discrete.

Notation: $G = PGL_2(\mathbb{R})$, $\Gamma = PGL_2(\mathbb{Z})$, $X = \Gamma \backslash G$,

$G_p = PGL_2(\mathbb{Q}_p)$, $K_p = PGL_2(\mathbb{Z}_p)$, $\Gamma_p = PGL_2(\mathbb{Z}[\frac{1}{p}])$.

discrete subgroup of $G \times G_p$.

Prop. $X \simeq \Gamma_p \backslash (G \times G_p) / K_p$ as G-spaces.

We claim that the action of G on $\Gamma_p \backslash (G \times G_p)/K_p$ is transitive. Equivalently, $G_p = \Gamma_p \cdot K_p$.

Given $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_p$, we show that

$$\exists \gamma \in \Gamma_p,\ k \in K_p :\ \gamma g k = I:$$

$$g \xrightarrow[\times \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}]{} \{|a|_p \geq |b|_p\} \xrightarrow[\times \begin{pmatrix} 1 & -\frac{b}{a} \\ 0 & 1 \end{pmatrix}]{} \begin{pmatrix} a & 0 \\ c & d' \end{pmatrix} \xrightarrow[\begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \times]{} \begin{pmatrix} a & 0 \\ c' & d' \end{pmatrix}$$

If $\alpha \in \mathbb{Z}[\frac{1}{p}]$, $\alpha \approx -\frac{c}{a}$, then $c' \approx 0$ and $|c'|_p \leq |d'|_p$.

$$\begin{pmatrix} a & 0 \\ c' & d' \end{pmatrix} \xrightarrow[\times \begin{pmatrix} 1 & 0 \\ -\frac{c'}{d'} & 1 \end{pmatrix}]{} \begin{pmatrix} a & 0 \\ 0 & d' \end{pmatrix}, \quad a, d' \in p^{\mathbb{Z}} \cdot \mathbb{Z}_p^{\times}.$$

This proves the claim.

Now $X \simeq \mathrm{Stab}_G(\Gamma_p (e,e) K_p) \backslash G$.

$$\Gamma_p (g,e) K_p = \Gamma (e,e) K_p \iff \exists \gamma \in \Gamma_p,\ k \in K_p : \begin{cases} \gamma g = e \\ \gamma k = e \end{cases}$$

$$\iff g \in \Gamma_p \cap K_p = \Gamma.$$

The group G_p is equipped with invariant measure m_p, which we normalise, so that $m_p(K_p)=1$.

<u>Def</u> For $f: X \to \mathbb{C}$, the <u>Hecke operator</u>

$$T_{p^n} f(\Gamma g) = \int_{B_n} f(\Gamma_p (g,b) K_p)\, dm_p(b),$$

where $B_n = K_p \begin{pmatrix} 1 & 0 \\ 0 & p^n \end{pmatrix} K_p$.

Properties: 1) T_{p^n} is self-adjoint,

2) T_{p^n} commutes with the action of G and D_X, $X \in \mathfrak{g}$.

A <u>lattice</u> in $\mathbb{Q}_p^2$ is $\mathbb{Z}_p v_1 + \mathbb{Z}_p v_2$ where $\{v_1, v_2\}$ is a basis of $\mathbb{Q}_p^2$.

$L_1 \sim L_2$ if $L_1 = \alpha L_2$ with $\alpha \in \mathbb{Q}_p^\times$.

$X_p = \{\text{equivalence classes of lattices in } \mathbb{Q}_p^2\}$.

Note that $G_p = PGL_2(\mathbb{Q}_p)$ acts transitively on X_p, and $\mathrm{Stab}_{G_p}(\mathbb{Z}_p e_1 + \mathbb{Z}_p e_2) = PGL_2(\mathbb{Z}_p) = K_p$. Hence,

$$X_p \simeq G_p / K_p .$$

<u>Lem.</u> (Cartan decomposition) $GL_2(\mathbb{Q}_p) = GL_2(\mathbb{Z}_p) \cdot \left\{ \begin{pmatrix} p^m & 0 \\ 0 & p^n \end{pmatrix} : m \leq n \right\} GL_2(\mathbb{Z}_p)$.

⌈ Similar to the previous proof: apply rows/column operations to reduce $g \in G_p$ to diagonal form. ⌋

<u>Cor.</u> Let L and M be lattices in $\mathbb{Q}_p^2$. Then $\exists$ a basis $\{v_1, v_2\}$ of L and integers $m \leq n$ such that $\{p^m \cdot v_1, p^n \cdot v_2\}$ is a basis of M.

⌈ Without loss of generality,
$L = \mathbb{Z}_p e_1 + \mathbb{Z}_p e_2$ and $M = g(\mathbb{Z}_p e_1 + \mathbb{Z}_p e_2)$ with $g \in GL_2(\mathbb{Q}_p)$.

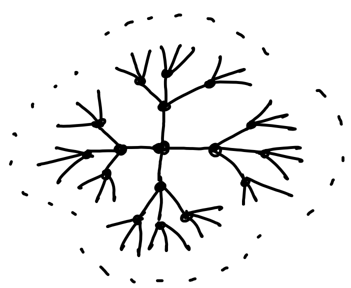
Write $g = k_1 \begin{pmatrix} p^m & 0 \\ 0 & p^n \end{pmatrix} k_2$ with $k_i \in GL_2(\mathbb{Z}_p)$.

Then $\{k_1^{-1} e_1, k_1^{-1} e_2\}$ and $\{k_2 e_1, k_2 e_2\}$ are the required bases. ⌋

Def. $[L] \neq [M] \in X_p$ are <u>neighbours</u> if for some representatives L, M, $pL \subset M \subset L$.

This defines a graph structure on X_p.

Prop. X_p is a $(p+1)$-regular tree.



Every neighbour of $[L]$ corresponds to a 1-dim. subspace of

$$L/pL \simeq \mathbb{F}_p \oplus \mathbb{F}_p.$$

Hence, there are $p+1$ neighbours.

The claim that $\nexists$ no loops follows from the following lemma.

Lem. Consider a chain of lattices $L_0 \supsetneq L_1 \supsetneq \dots \supsetneq L_n$ such that $L_{i+1} \supsetneq pL_i$, $pL_{i-1} \neq L_{i+1}$.
Then there exists a basis $\{v_1, v_2\}$ of L_0 such that $L_i = \mathbb{Z}_p v_1 + \mathbb{Z}_p (p^i v_2)$.

We argue by induction on n.

Suppose that $L_i = \mathbb{Z}_p v_1 + \mathbb{Z}_p(p^i v_2)$ for $i<n$.

The lattice L_n corresponds to a 1-dim subspace of $L_{n-1}/pL_{n-1} \simeq \mathbb{F}_p \oplus \mathbb{F}_p$.

Since $L_n \neq pL_{n-2} = \mathbb{Z}_p(pv_1) + \mathbb{Z}_p(p^{n-1}v_2)$,

this subspace is not $\langle (0,1) \rangle$.

Hence, $L_n = \underbrace{\langle v_1 + x\cdot p^{n-1} v_2, \; pL_{n-1} \rangle}_{\langle v_1 + x\cdot p^{n-1} v_2, \; p^n v_2 \rangle}$ for some $x = 0, \dots p-1$.

Then $\{ v_1 + xp^{n-1} v_2, \; v_2 \}$ is a required basis.

In particular, it follows that if $d([L], [\mathbb{Z}_p e_1 + \mathbb{Z}_p e_2]) = n$, then $[L] = \left[k \begin{pmatrix} 1 & 0 \\ 0 & p^n \end{pmatrix} (\mathbb{Z}_p e_1 + \mathbb{Z}_p e_2) \right]$ for $k \in K_p$.

<u>Notation:</u> For $x = \Gamma g \in X$,

$$\underset{\cup}{\Delta_p(x)} = \Gamma_p \backslash (\{g\} \times X_p) \text{ — } \underline{\text{Hecke tree}} \text{ through } x$$

$$\Delta_p^{(n)}(x) \text{ — vertices with distance } n \text{ from } x.$$

Then
$$\boxed{T_{p^n} f(x) = \sum_{y \in \Delta_p^{(n)}(x)} f(y)}$$

This formula implies that:

<u>Lem.</u> 1) $T_p^2 = T_{p^2} + (p+1) I$,

2) $T_p \cdot T_{p^n} = T_{p^{n+1}} + p\, T_{p^{n-1}}$, $n \geq 2$.

<u>Thm.</u> Let f be a function on a tree such that $T_p f = \lambda f$. Then $\exists c > 0$ (absolute constant):
$$\sum_{y : d(x,y) \leq n} |f(y)|^2 \geq c \cdot n\, |f(x)|^2.$$

By Cauchy-Schwarz inequality,
$$\left| \sum_{i=0}^{n} T_{p^i} f(x) \right| = \left| \sum_{y : d(x,y) \leq n} f(y) \right| \leq \left(\sum_{y : d(x,y) \leq n} |f(y)|^2 \right)^{1/2} \cdot \#\{y : d(x,y) \leq n\}^{1/2}$$
$$\leq \text{const} \cdot p^{n/2} \cdot \left(\sum_{y : d(x,y) \leq n} |f(y)|^2 \right)^{1/2}$$

It follows from the lemma that f is also an eigenvector of T_{p^i} and the eigenvalues λ_i satisfy the 2nd order recurrence relation:

$$\lambda_{i+1} = \lambda \cdot \lambda_i - p \cdot \lambda_{i-1}, \quad i \geq 2,$$
$$\lambda_1 = \lambda, \quad \lambda_2 = \lambda^2 - p - 1.$$

Hence, $\lambda_i = \alpha \cdot u^i + \beta \cdot v^i$

where u & v are solutions of $x^2 - \lambda x + p = 0$.

It is sufficient to show that

$$\Big| \underbrace{\sum_{i=0}^{n} \lambda_i}_{} \Big| \geq \text{const} \cdot p^{n/2} \cdot n.$$

This can be computed explicitly as a geometric series.

We omit details...

Def. A measure ν on X is called Hecke recurrent if for every measurable $B \subset X$ and ν-almost every $x \in B$,

$$\Delta_p^{(n)}(x) \cap B \neq \emptyset \text{ for infinitely many } n.$$

Thm. Let φ_k be a sequence of eigenfunctions of T_p, $\|\varphi_k\|_2 = 1$, and $d\nu_k = |\varphi_k(g)|^2 d\mu(g)$. Then every limit ν of ν_k's is Hecke recurrent.

Γ Since T_{p^i} are self-adjoint, it follows from the Theorem that for $f \in C_c(X)$, $f \geq 0$:

$$\left\langle \sum_{i=0}^{n} T_{p^i} f, |\varphi_k|^2 \right\rangle = \left\langle f, \sum_{i=0}^{n} T_{p^i} |\varphi_k|^2 \right\rangle$$

$$\geq \text{const} \cdot n \cdot \langle f, |\varphi_k| \rangle.$$

Passing to the limit, we obtain

$$\int_X \left(\sum_{i=0}^{n} T_{p^i} f \right) d\nu \geq \text{const} \cdot n \cdot \int_X f. \qquad (*)$$

This extends to general non-negative measurable functions.

Let $B \subset X$ be a measurable set.

Let $B_k = \{x \in B : B \cap \Delta_\varphi^{(k)}(x) = \emptyset\}$ and $C_\ell = \bigcap_{k \geq \ell} B_k$.

The set of points in B which does not come back to B infinitely often is $\bigcup_{\ell \geq 1} C_\ell$.

We apply $(*)$ to $f = \chi_{C_\ell}$.

For every $z \in X$, $\Delta_\varphi(z) \cap C_\ell$ has diameter $\leq \ell$ on the tree $\Delta_\varphi(z)$. Then $\sum_{i=0}^{\infty} T_{p^i} f$ is bounded, and by $(*)$, $\nu(C_\ell) = 0$.

Hence, for ν-a.e. $x \in B$, $B \cap \Delta_\varphi^{(n)}(x) \neq \emptyset$ for infinitely many n. $\square$