Lecture 9

Entropy of quantum limits.

$M = \Gamma \backslash \mathbb{H}$ – finite area hyperbolic surface,
Γ – arithmetic lattice (e.g., $\Gamma = PGL_2(\mathbb{Z})$).

φ_λ – eigenfunction of Δ, $\|\varphi_\lambda\|_2 = 1$,
which is also an eigenfunction of T_p's.

$d\mu_\lambda = |\varphi_\lambda|^2 d\mu$ – prob. measure.

Conj (arithmetic quantum unique ergodicity)

$$\boxed{\mu_\lambda \xrightarrow[\lambda\to\infty]{} \mu}$$

This was proved by Lindenstrauss
(modulo divergence issue, for noncompact surfaces)
and Soundararajan (nondivergence).

The microlocal lift gives measures $d\nu_\lambda = |\psi_\lambda|^2 dm$
on $X = \Gamma \backslash G$ which project to μ_λ.

Since T_p commutes with the actions of G and D_X, $X \in \mathfrak{g}$, ψ_λ is also an eigenfunction of T_p.

Let ν = a limit of ν_λ as $\lambda \to \infty$.

Assume that ν is a probability measure.

We know: – ν is g^t-invariant,

– ν is T_p-recurrent.

Def. A Bowen ball:

$$B_{n,\delta}(x) = \{ y \in X : \; d(g^i(x), g^i(y)) < \delta \}$$
$$i = -n, \dots, n.$$

$$\nu(B_{n,\delta}(x)) \text{ - small} \iff g^t \circlearrowright (X, \nu) \text{ is chaotic.}$$

Bourgain – Lindenstrauss

Thm (positive entropy) $\exists h > 0$: for ν-a.e. x, and infinitely many n,

$$\boxed{\nu(B_{n,\delta}) \leq c(x,\delta) \cdot e^{-hn},} \qquad (E)$$

uniformly on x in compact sets.

Lem. $\exists$ uniform $c>0$: $\forall f \geq 0$:

$$\int_X \left(T_p(f) + T_{p^2}(f)\right) d\nu \geq c \cdot \int_X f \, d\nu .$$

Recall that $T_{p^2} = T_p^2 - p - 1$.

If $T_p \psi = \lambda \psi$, then $T_{p^2} \psi = \lambda_2 \psi$, $\lambda_2 = \lambda^2 - p - 1$.

Hence, either $|\lambda| \geq \frac{1}{2}\sqrt{p}$ or $|\lambda_2| \geq \frac{1}{2} p$.

Suppose, for instance, that $|\lambda| \geq \frac{1}{2}\sqrt{p}$.

Then by Cauchy-Schwartz inequality,

$$\frac{1}{2}\sqrt{p}\, |\psi(x)| \leq |T_p \psi(x)| \leq \left(\sum_{y \in \Delta_p^{(1)}(x)} |\psi(y)|^2 \right)^{1/2} \cdot \sqrt{p+1} .$$

Similar argument also applies in the other case, and obtain $T_p(|\psi|^2) + T_{p^2}(|\psi|^2) \geq c \cdot |\psi|^2$.

Finally, $\int_X \left(T_p(f) + T_{p^2}(f)\right) d\nu_\lambda = \langle T_p f + T_{p^2} f, |\psi_\lambda|^2 \rangle$

$= \langle f, T_p(|\psi_\lambda|^2) + T_{p^2}(|\psi_\lambda|^2) \rangle \geq c \cdot \int_X f \, d\nu_\lambda$,

and passing to limit as $\lambda \to \infty$, we deduce the lemma.

<u>Proof of Thm.</u> 1) <u>Recurrence along Hecke tree.</u>

Let $B_n = B_{n,\delta}(x) = x\cdot\underbrace{\left(\bigcap_{i=-n}^{n} a^i G_\delta a^{-i}\right)}_{G_{n,\delta}}$, where

$a = \begin{pmatrix} e^{1/2} & 0 \\ 0 & e^{-1/2}\end{pmatrix}$ and G_δ is a δ-nbhd of identity in G.

We suppose that $\boxed{\nu(B_n) \geq e^{-hn}}$, $h \approx 0$, $(+)$

for all sufficiently large n.

By Lemma, $\displaystyle\int_X \sum_{p\leq Q}\left(T_p(\chi_{B_n}) + T_{p^2}(\chi_{B_n})\right)d\nu \geq c\cdot\frac{Q}{\log Q}\cdot|B_n|.$

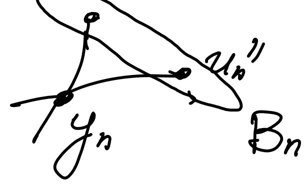
We pick $Q_n \approx e^{2hn}$, so that

$$\int_X \sum_{p\leq Q_n}\left(T_p(\chi_{B_n}) + T_{p^2}(\chi_{B_n})\right)d\nu > 1,$$

and $\displaystyle\sum_{p\leq Q_n}\left(\sum_{y\in\Delta_p^{(1)}(y_n)\cup\Delta_p^{(2)}(y_n)} \chi_{B_n}(y)\right) > 1$ for some $y_n \in X$.

At least <u>two</u> terms in the sum are $\neq 0$.

For simplicity, let's say



$\exists\, u_n' \neq u_n'' \in \Delta_p^{(1)}(y_n) \cap B_n.$

Write $y_n = \Gamma g_n$. Then $\Delta_p^{(1)}(y_n) = \Gamma_p(g_n, K_p\left(\begin{smallmatrix}1&0\\0&p\end{smallmatrix}\right)K_p)/K_p$.

Since $\Gamma = PGL_2(\mathbb{Z})$ is dense in $K_p = PGL_2(\mathbb{Z}_p)$,
the cosets $K_p\left(\begin{smallmatrix}1&0\\0&p\end{smallmatrix}\right)K_p/K_p$ can be represented by
$\{\gamma_i\} \subset \Gamma\left(\begin{smallmatrix}1&0\\0&p\end{smallmatrix}\right)\Gamma$, and $\Gamma_p(g_n, K_p\left(\begin{smallmatrix}1&0\\0&p\end{smallmatrix}\right)K_p)/K_p = \{\Gamma_p(g_n,\gamma_i)K_p\}$
$= \{\Gamma_p(\gamma_i^{-1}g_n, e)K_p\}$.

Hence, $\exists\, \gamma', \gamma'' \in \Gamma\left(\begin{smallmatrix}1&0\\0&p\end{smallmatrix}\right)\Gamma$: $u_n' = \Gamma(\gamma')^{-1}g_n$, $u_n'' = \Gamma(\gamma'')^{-1}g_n$.
$\gamma'\Gamma \neq \gamma''\Gamma$.

Write $x = \Gamma g$. Then $u_n' = \Gamma g b'$, $u_n'' = \Gamma g b''$
for $b', b'' \in G_{n,\delta}$.

Then for some $\gamma', \gamma'' \in \Gamma\left(\begin{smallmatrix}1&0\\0&p\end{smallmatrix}\right)\Gamma$, $\gamma'\Gamma \neq \gamma''\Gamma$:

$$\begin{cases}(\gamma')^{-1}g_n = g b', \\ (\gamma'')^{-1}g_n = g b''.\end{cases} \Longrightarrow \underbrace{(\gamma')^{-1}\gamma''}_{\eta_n} = g\underbrace{b'(b'')^{-1}}_{b}g^{-1}$$

where $\eta_n \in SL_2(\mathbb{Q})$ with denominator $\leq p$, $\eta_n \neq I$

$$b \in G_{n,\delta}\cdot G_{n,\delta}^{-1} \subset G_{n,2\delta},$$

$$\boxed{g^{-1}\eta_n g \in G_{n,2\delta}} \qquad (*)$$

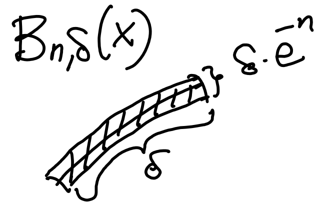
2) Structure of Bowen balls.

For $b \in G_\delta$, $b = \exp(tH + \alpha U^+ + \beta U^-)$ with $|t|, |\alpha|, |\beta| \ll \delta$,

and $a^i \cdot b \cdot a^{-i} = \exp(a^i(tH + \alpha U^+ + \beta U^-)a^{-i})$

$$= \exp(tH + (\alpha \cdot e^{-i})U^+ + (\beta \cdot e^i)U^-).$$

Hence, $G_{n,\delta} = \bigcap_{i=-n}^{n} a^i G_\delta a^{-i}$ is $(\delta \cdot e^{-n})$-neighbourhood of piece of diagonal.



For $g_1, g_2 \in G_{n,\delta}$, $g_i = a_i v_i$ with diagonal $a_i = I + O(\delta)$, $v_i = I + O(\delta \cdot e^{-n})$.

Then
$$[g_1, g_2] = (a_1 v_1)^{-1}(a_2 v_2)^{-1}(a_1 v_1)(a_2 v_2)$$
$$= v_1^{-1} a_1^{-1} v_2^{-1} a_2^{-1} a_1 v_1 a_2 v_2$$
$$= v_1^{-1} \underbrace{(a_1^{-1} v_2^{-1} a_1)}_{I+O(\delta e^{-n})} \cdot \underbrace{(a_2^{-1} v_1 a_2)}_{I+O(\delta e^{-n})} \cdot v_2$$
$$= I + O(\delta e^{-n}).$$

Now we apply this estimate to η_n's.

3) Classification of η_n's.

Then $[\eta_n, \eta_{n+1}] = I + O(\delta \cdot e^{-n})$.

On the other hand, $[\eta_n, \eta_{n+1}]$ is rational with denominator $\leq p^4 \leq e^{8hn}$, $h \approx 0$.

This implies that $[\eta_n, \eta_{n+1}] = I$ for sufficiently large n.

Now we claim that η_n's are hyperbolic ($\Leftrightarrow |Tr(\eta_n)| > 2$).

Since $g^{-1}\eta_n g = a_n v_n$ with $a_n = \begin{pmatrix} e^{t_n/2} & 0 \\ 0 & e^{-t_n/2} \end{pmatrix}$,
$v_n = I + O(\delta e^{-n})$,

$Tr(\eta_n) = e^{t_n/2} + e^{-t_n/2} + O(\delta e^{-n})$.

If $|Tr(\eta_n)| \leq 2$, then $e^{t_n/2} + e^{-t_n/2} \leq 2 + O(\delta e^{-n})$,
and $t_n = O(\delta e^{-n})$.

Then $\eta_n = I + O(\delta \cdot e^{-n})$. Since the denominator of η_n is $\geq e^{-2hn}$, $h \approx 0$, $\eta_n = I$, which is a contradiction.

Hence, all γ_n's are hyperbolic and commuting.

In particular, $g^{-1}\gamma_n g \in A$ – a fixed diagonilisable subgroup.

Write $g^{-1}\gamma_n g = \exp(t_n X)$ with fixed $\|X\| = 1$.

Since $g^{-1}\gamma_n g \in G_{n,\delta}$, $t_n X = t_n' H + O(e^{-n})$, $t_n' \geq e^{-n/2}$.

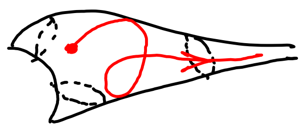
This implies that $X = H$ and A is the diagonal subgroup.

1) Suppose that $\gamma = \gamma_n$ is <u>diagonalisable over $\mathbb{Q}$.</u>

$\exists h \in SL_2(\mathbb{R})$ proportional to a rational matrix: $h^{-1}\gamma h \in A$.

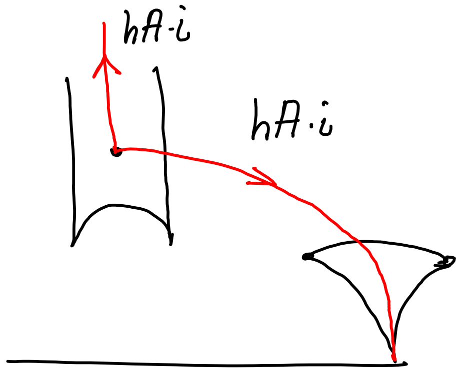
Then $g \in h \cdot N_{SL_2(\mathbb{R})}(A) = hA \cup h\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} A$,

so $x = \Gamma g \in \Gamma h A \cup \Gamma h \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} A$.



The end points of the geodesic $h \cdot A \cdot i$ (and $\Gamma h \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} A$) are rational.

This implies that diverges (i.e., escapes every compact set in G/Γ).



By Poincare recurrence, $\nu(\Gamma h A)=0$, and x belongs to the set zero measure:

$$\bigcup_h \left(\Gamma h A \cup \Gamma h \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} A\right).$$

2) Suppose that $\eta = \eta_n$ is <u>not diagonalisable</u> $/\mathbb{Q}$ (i.e., the characteristic polynomial of η is irreducible). Then $K = \mathbb{Q}[\eta] \subset M_2(\mathbb{Q})$ is a quadratic field. Let $\mathcal{O} = \mathbb{Z}[\eta]$ and $\mathcal{O}^\times$ is the group of invertable elements. Since $\eta \in M_2(\mathbb{Q})$, $\underbrace{\mathcal{O} \cap M_2(\mathbb{Z})}_{\mathcal{U}}$ has finite index in $\mathcal{O}$.

<u>Facts:</u> 1) $\mathcal{U}^\times$ has finite index in $\mathcal{O}^\times$,

2) $\exists$ primes p: $\mathcal{U}^\times \cdot p^{\mathbb{Z}}$ has finite index in $\mathcal{O}[\frac{1}{p}]^\times$.

Recall that $\eta \cdot g = g \cdot a$ for some $a \in A$.

By 1), $\eta^k \in \mathcal{U}^\times \subset Gl_2(\mathbb{Z})$ for some $k \in \mathbb{N}$.

Then $\Gamma g a^k = \Gamma \eta^k g = \Gamma g$, i.e. $x = \Gamma g$ is contained in a periodic orbit xA.

We claim that $\nu(xA) = 0$.

Suppose not. Then we apply p-Hecke recurrence:

$\exists y \in xA: y_n \in \Delta_p(y): y_n \neq y_m$ for $n \neq m$, $y_n \in xA$.

Multiplying by $a \in A$, we may assume that $y = x$.

As in step 1, $y_n = \Gamma \gamma_n^{-1} g$ for $\gamma_n = PGL_2(\mathbb{Z}[\frac{1}{p}])$,

$$\gamma_n K_p \neq \gamma_m K_p \text{ for } n \neq m.$$

As above, we have $\gamma_n^{-1} g = g a_n$ for some $a_n \in A$.

Then $\gamma_n^{-1} \in K$ and $\gamma_n^{-1} \in \mathcal{O}[\frac{1}{p}]^\times$.

By 2), $\mathcal{U}^\times p^{\mathbb{Z}}$ has finite index in $\mathcal{O}[\frac{1}{p}]^\times$.

Hence, $\exists n \neq m: \ \mathcal{U}^\times p^{\mathbb{Z}} \cdot \gamma_n^{-1} = \mathcal{U}^\times p^{\mathbb{Z}} \cdot \gamma_m^{-1}$.

Since $\mathcal{U}^\times \subset GL_2(\mathbb{Z})$, this implies that

$$PGL_2(\mathbb{Z}_p)\gamma_n^{-1} \neq PGL_2(\mathbb{Z}_p)\gamma_m^{-1},$$

which is a contradiction. Hence, $\nu(xA) = 0$.

Assuming (+), we proved that x belongs either to divergent A-orbit or periodic A-orbit. The union of such sets have ν-measure 0. $\square$

Now the proof of A.Q.U.E. reduces to the problem of classification of measures:

Lindenstrauss

Thm. (measure rigidity)

Let ν be a prob. measure on X such that:

1) ν is invariant under a geodesic flow,

2) ν is p-Hecke recurrent for some p,

3) ν has "positive entropy" (condition E).

Then ν is the invariant measure.