# LECTURE 11: ARITHMETIC FUNCTIONS

Recall that a function $f : \mathbb{N} \to \mathbb{C}$ is called an arithmetical function. Recall also that a multiplicative function $f$ satisfies the property that whenever $(m, n) = 1$, one has

$$f(mn) = f(m)f(n).$$

We have proved that the function

$$g(n) = \sum_{d|n} f(d)$$

is multiplicative whenever $f(n)$ is multiplicative. In this section we discuss various properties of arithmetical functions, many of them multiplicative, and seek to understand what they "look" like.

## 1. Examples of arithmetical functions

- the *divisor function* $\tau(n)$.
  The function $\tau(n)$ is defined as the number of divisors of the number $n$. It can be written explicitly as
  $$\tau(n) = \sum_{d|n} 1.$$
- the *sum of divisors function* $\sigma(n)$.
  The function $\sigma(n)$ is defined as the sum of all divisors of the number $n$. It can be written explicitly as
  $$\sigma(n) = \sum_{d|n} d.$$
- the *Euler totient function* $\phi(n)$.
  The function $\phi(n)$ is defined as the number of reduced residue classes modulo $n$. We have shown that it is given by a formula
  $$\phi(n) = \prod_{p^r \| n} (p^r - p^{r-1}).$$
- the *Möbius function* $\mu(n)$
  The function $\phi(n)$ is defined by
  $$\mu(n) = \begin{cases} (-1)^\ell, & \text{when } n = p_1 \cdots p_\ell \text{ is squarefree,} \\ 0, & \text{otherwise.} \end{cases}$$
  Here, by a *squarefree number*, we mean an integer that is not divisible by the square of any prime number. It is easy to check that $\mu$ is multiplicative.

Note that, just as with our earlier discussion of the Euler totient, a function that is multiplicative will be relatively easy to evaluate when its argument has a known prime factorisation. For example, one can see rather easily that when $p$ is a prime number and $h$ is a non-negative integer, then

$$\tau(p^r) = r + 1 \quad \text{and} \quad \sigma(p^r) = \sum_{i=0}^{h} p^i = \frac{p^{r+1} - 1}{p - 1},$$

and thus

$$\tau(n) = \prod_{p^r \| n} (r + 1) \quad \text{and} \quad \sigma(n) = \prod_{p^r \| n} \left( \frac{p^{r+1} - 1}{p - 1} \right).$$

## 2. The Möbius inversion formula

The Möbius function, has special properties that make it particularly useful in studying averages of other arithmetic functions (and much else besides). Recall that

$$\mu(n) = \begin{cases} (-1)^\ell, & \text{when } n = p_1 \dots p_\ell \text{ is squarefree,} \\ 0, & \text{otherwise.} \end{cases}$$

We define a rather trivial multiplicative function $\nu(n)$ by

$$\nu(n) = \begin{cases} 1, & \text{when } n = 1, \\ 0, & \text{otherwise.} \end{cases}$$

**Lemma 2.1.** *One has* $\sum_{d|n} \mu(d) = \nu(n)$.

*Proof.* Since $\mu(n)$ is a multiplicative function of $n$, it follows that $\sum_{d|n} \mu(d)$ is also multiplicative. But on writing $f(n) = \sum_{d|n} \mu(d)$, one finds that

$$f(p^\alpha) = \sum_{h=0}^{\alpha} \mu(p^h) = 1 - 1 = 0, \quad \text{for } \alpha > 0,$$

and $f(1) = \mu(1) = 1$. Thus, in view of the multiplicativity of $f(n)$, one finds that $f(n)$ is zero unless $n$ has no prime divisors, a circumstance that occurs only when $n = 1$. This completes the proof of the theorem. □

We can now describe a certain duality between arithmetic functions, and functions defined via divisor sums.

**Theorem 2.2** (the Möbius inversion formulae)**.** *(i) Let $f$ be any arithmetical function, and define*

$$g(n) = \sum_{d|n} f(d).$$

*Then one has*

$$f(n) = \sum_{d|n} \mu(d) g(n/d).$$

*(ii) Suppose that g is any arithmetical function, and define*

$$f(n) = \sum_{d|n} \mu(d) g(n/d).$$

*Then one has*

$$g(n) = \sum_{d|n} f(d).$$

*Proof.* (i) Given that $g(n) = \sum_{d|n} f(d)$, one obtains

$$\begin{aligned}\sum_{d|n} \mu(d) g(n/d) &= \sum_{d|n} \sum_{e|(n/d)} \mu(d) f(e) = \sum_{e|n} f(e) \sum_{d|(n/e)} \mu(d) \\ &= \sum_{e|n} f(e) \nu(n/e) = f(n).\end{aligned}$$

(ii) Given that $f(n) = \sum_{d|n} \mu(d) g(n/d)$, one obtains

$$\begin{aligned}\sum_{d|n} f(d) = \sum_{d|n} f(n/d) &= \sum_{d|n} \sum_{e|(n/d)} \mu(e) g(n/(de)) \\ &= \sum_{e|n} \sum_{d|(n/e)} \mu(e) g(n/(de)) = \sum_{e|n} \sum_{d|(n/e)} \mu(e) g(d) \\ &= \sum_{d|n} g(d) \sum_{e|(n/d)} \mu(e) = \sum_{d|n} g(d) \nu(n/d) = g(n).\end{aligned}$$

□

Note that Möbius inversion applies to all arithmetical functions, without any hypothesis concerning whether or not they are multiplicative.

**Example 2.3.** Recall that we showed that

$$\sum_{d|n} \phi(d) = n.$$

As an immediate consequence of the Möbius inversion formulae, we deduce that

$$\phi(n) = \sum_{d|n} \mu(d)(n/d) = n \sum_{d|n} \mu(d)/d.$$

## 3. Estimates for arithmetical functions

We now explore the "population statistics" of values of arithmetical functions: what is the maximal/minimal size of such a function, the average size, the variance, etc.? In order properly to discuss such issues, we need to recall some standard analytic notation.

Given functions $f, g : \mathbb{R} \to \mathbb{R}$, with $g$ taking positive values, we write

$$f(x) = O(g(x)) \quad (\text{for } x \geqslant x_0)$$

when there exists some positive constant $C$ for which

$$|f(x)| \leqslant Cg(x) \quad (\text{for } x \geqslant x_0).$$

**Example 3.1.** One has $x = O(x^2)$ for $x \geqslant 1$, $1/x^2 = O(1)$ for $x \geqslant 1$, and $x = O(e^x)$ for $x \geqslant 0$.

There are two useful strategies to keep in mind when addressing questions concerning estimates for arithmetic functions:

- In order to estimate the *size* of a *multiplicative* function $f(n)$, one should first estimate $f(p^r)$ on prime powers, and then combine this information with knowledge about the distribution of prime numbers.
- If one wishes to estimate the average size of an arithmetical function $g(n)$, one can apply the Möbius inversion formulae to write $g(n)$ in the shape

  $$g(n) = \sum_{d|n} f(d),$$

  in which

  $$f(n) = \sum_{d|n} \mu(d) g(n/d).$$

  Frequently, one finds that this new function $f(n)$ is reasonably well behaved, and then one has

  $$\begin{aligned}\sum_{1\leqslant n\leqslant x} g(n) &= \sum_{1\leqslant n\leqslant x} \sum_{d|n} f(d) \\ &= \sum_{1\leqslant d\leqslant x} \sum_{1\leqslant m\leqslant x/d} f(d).\end{aligned}$$

  Here, in the last summation, we made the change of variable $n = md$. Thus we obtain

  $$\begin{aligned}\sum_{1\leqslant n\leqslant x} g(n) &= \sum_{1\leqslant d\leqslant x} f(d) \sum_{1\leqslant m\leqslant x/d} 1 \\ &= \sum_{1\leqslant d\leqslant x} f(d) \left\lfloor \frac{x}{d} \right\rfloor,\end{aligned}$$

  where, as usual, we write $\lfloor \theta \rfloor$ for the greatest integer not exceeding $\theta$. Thus we see that

  $$\begin{aligned}\frac{1}{x} \sum_{1\leqslant n\leqslant x} g(n) &= \frac{1}{x} \sum_{1\leqslant d\leqslant x} f(d) \left( \frac{x}{d} + O(1) \right) \\ &= \sum_{1\leqslant d\leqslant x} \frac{f(d)}{d} + O\left( \frac{1}{x} \sum_{1\leqslant d\leqslant x} |f(d)| \right).\end{aligned}$$

  In many circumstances, the first term on the right hand side of the last equation is of larger order of magnitude than the last term, and then one has the asymptotic formula

$$\frac{1}{x}\sum_{1\leqslant n\leqslant x} g(n) \sim \sum_{1\leqslant d\leqslant x}\frac{f(d)}{d}.$$

This formula is useful provided that the new average is easier to compute than the original average.

We illustrate these ideas with some examples.

**Example 3.2.** The divisor function $\tau(n)$.

We claim that for any positive number $\varepsilon$, one has

$$\tau(n) = O(n^{\varepsilon}) \quad \text{for } n \in \mathbb{N}.$$

In order to establish this estimate, we exploit the multiplicative property of $\tau(n)$, and investigate the function

$$\frac{\tau(n)}{n^{\varepsilon}} = \prod_{p^j \| n}\frac{j+1}{p^{j\varepsilon}}.$$

For any fixed prime $p$, $(j+1)/p^{j\varepsilon}$ is a decreasing function of $j$ for sufficiently large $j$. In fact, with a little calculus we find that

$$\frac{j+1}{p^{j\varepsilon}} = \frac{p^{\varepsilon}}{\varepsilon\log_2 p}\frac{(j+1)\varepsilon\log_2 p}{2^{(j+1)\varepsilon\log_2 p}} \leqslant \frac{p^{\varepsilon}}{\varepsilon\log_2 p}\sup\{x2^{-x} : x \geqslant 0\} = O(p^{\varepsilon}).$$

Thus,

$$\frac{\tau(n)}{n^{\varepsilon}} \leqslant \prod_{p^j\|n} O(p^{\varepsilon}) = O(n^{\varepsilon}).$$

Next we turn to the matter of the average value of $\tau(n)$ for $1 \leqslant n \leqslant x$. In this instance, of course, one has $\tau(n) = \sum_{d|n} 1$, and so

$$\begin{aligned}\sum_{1\leqslant n\leqslant x}\tau(n) &= \sum_{1\leqslant n\leqslant x}\sum_{d|n} 1 = \sum_{1\leqslant d\leqslant x}\sum_{1\leqslant m\leqslant x/d} 1 \\ &= \sum_{1\leqslant d\leqslant x}\lfloor x/d\rfloor = \sum_{1\leqslant d\leqslant x}(x/d + O(1)) \\ &= x\sum_{1\leqslant d\leqslant x}\frac{1}{d} + O(x).\end{aligned}$$

But (as a good exercise in calculus),

$$\begin{aligned}\sum_{1\leqslant d\leqslant x}\frac{1}{d} &= \sum_{1\leqslant d\leqslant x}\left(\int_d^{d+1}\frac{dt}{t} + O(1/d^2)\right) \\ &= \log x + O(1).\end{aligned}$$

Thus we deduce that

$$\frac{1}{x}\sum_{1\leqslant n\leqslant x}\tau(n) = \log x + O(1).$$

**Example 3.3.** The sum of divisors function $\sigma(n)$.

In this instance we have the formula $\sigma(n) = \sum_{d|n} d$, and so

$$\sum_{1\leqslant n\leqslant x} \sigma(n) = \sum_{1\leqslant n\leqslant x} \sum_{d|n} \frac{n}{d} = \sum_{1\leqslant d\leqslant x} \sum_{1\leqslant m\leqslant x/d} m$$
$$= \sum_{1\leqslant d\leqslant x} \tfrac{1}{2}\lfloor x/d\rfloor(\lfloor x/d\rfloor + 1) = \sum_{1\leqslant d\leqslant x} \left(\frac{x^2}{2d^2} + O(x/d)\right).$$

But

$$\sum_{1\leqslant d\leqslant x} \frac{1}{d^2} = \sum_{d=1}^{\infty} \frac{1}{d^2} + O\left(\sum_{d>x} 1/d^2\right) = \zeta(2) + O(1/x),$$

and hence

$$\frac{1}{x}\sum_{1\leqslant n\leqslant x} \sigma(n) = \tfrac{1}{2}\zeta(2)x + O(\log x) = \frac{\pi^2}{12}x + O(\log x).$$

**Example 3.4.** The Euler function $\phi(n)$.

We shall use the formula

$$\phi(n) = n\sum_{d|n} \frac{\mu(d)}{d}.$$

Following the above strategy, we find that

$$\sum_{1\leqslant n\leqslant x} \phi(n) = \sum_{1\leqslant n\leqslant x} \sum_{d|n} \mu(d)n/d = \sum_{1\leqslant d\leqslant x} \mu(d) \sum_{1\leqslant m\leqslant x/d} m$$
$$= \sum_{1\leqslant d\leqslant x} \mu(d) \cdot \tfrac{1}{2}\lfloor x/d\rfloor(\lfloor x/d\rfloor + 1) = \sum_{1\leqslant d\leqslant x} \mu(d)\left(\frac{x^2}{2d^2} + O(x/d)\right)$$
$$= \tfrac{1}{2}x^2 \sum_{1\leqslant d\leqslant x} \frac{\mu(d)}{d^2} + O\left(\sum_{1\leqslant d\leqslant x} |\mu(d)|x/d\right).$$

To estimate the error term, we observe that

$$\sum_{1\leqslant d\leqslant x} \frac{1}{d} < 1 + \int_1^x \frac{dt}{t} = O(\log x).$$

Furthermore,

$$\sum_{1\leqslant d\leqslant x} \frac{\mu(d)}{d^2} = \sum_{d=1}^{\infty} \frac{\mu(d)}{d^2} + O\left(\sum_{d>x} 1/d^2\right) = \sum_{d=1}^{\infty} \frac{\mu(d)}{d^2} + O(1/x).$$

Thus

$$\frac{1}{x}\sum_{1\leqslant n\leqslant x} \phi(n) = Cx + O(\log x),$$

where $C = \sum_{d=1}^{\infty} \frac{\mu(d)}{d^2}$. In fact, one can show that $C = 6/\pi^2$. In some sense, this means that $\phi(n)$ is, on average, about $(6/\pi^2)n$.

**Example 3.5.** The number of squarefree numbers.

We turn our attention next to counting the number of squarefree numbers up to $x$. Define

$$S(x) = \#\{1 \leqslant n \leqslant x : n \text{ is squarefree}\}.$$

In order to analyse this sum, we need to have available a detector for squarefree numbers. If we recall that the sum of the Möbius function over the divisors of an integer $n$, which we called $\nu(n)$, is non-zero precisely when $n = 1$, in which case it is 1, we are led to consider the expression

$$\sum_{d^2 \mid n} \mu(d).$$

Let $m$ be the largest positive integer with $m^2 \mid n$. Then the above expression is

$$\sum_{d \mid m} \mu(d) = \nu(m) = \begin{cases} 1, & \text{when } m \text{ is equal to 1, i.e. } n \text{ is squarefree,} \\ 0, & \text{when } m > 1, \text{ i.e. } n \text{ is not squarefree.} \end{cases}$$

Thus we find that

$$S(x) = \sum_{1 \leqslant n \leqslant x} \sum_{d^2 \mid n} \mu(d).$$

But this expression has similar shape to those that we have considered before in this section, and so we may analyse this sum similarly. By means of the change of variable $n = md^2$, one finds that

$$\begin{aligned} S(x) &= \sum_{1 \leqslant d \leqslant \sqrt{x}} \sum_{1 \leqslant m \leqslant x/d^2} \mu(d) = \sum_{1 \leqslant d \leqslant \sqrt{x}} \mu(d) \lfloor x/d^2 \rfloor \\ &= \sum_{1 \leqslant d \leqslant \sqrt{x}} \mu(d) \left( \frac{x}{d^2} + O(1) \right) = x \sum_{1 \leqslant d \leqslant \sqrt{x}} \frac{\mu(d)}{d^2} + O\left( \sum_{1 \leqslant d \leqslant \sqrt{x}} 1 \right). \end{aligned}$$

But

$$\sum_{1 \leqslant d \leqslant \sqrt{x}} \frac{\mu(d)}{d^2} = \sum_{d=1}^{\infty} \frac{\mu(d)}{d^2} + O\left( \sum_{d > \sqrt{x}} \frac{1}{d^2} \right) = C + O(1/\sqrt{x}).$$

Thus we conclude that

$$\begin{aligned} S(x) &= x \left( C + O(1/\sqrt{x}) \right) + O(\sqrt{x}) \\ &= \frac{6}{\pi^2} x + O(\sqrt{x}). \end{aligned}$$

Thus the "probability" that a randomly chosen positive integer is squarefree is $6/\pi^2$.