Rational points on algebraic varieties and homogeneous flows

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Basic questions

Consider a system of polynomial equations

$X = \{f_1(x_0, \ldots, x_d) = \cdots = f_k(x_0, \ldots, x_d) = 0\}.$

Question
Is the set $X(\mathbb{Q})$ infinite or finite?

Question
Is the set $X(\mathbb{Q})$ dense in $\left(\begin{array}{l} \textit{Zariski topology} \\ \textit{Euclidean topology} \end{array} \right)$*?*

We assume, first, that f_i's are homogeneous.

Then

$$X(\mathbb{Q}) \subset \mathbb{P}^d(\mathbb{Q}) = \text{projective space.}$$

For projective varieties, the relation

$$\textit{geometry} \quad \longleftrightarrow \quad \textit{arithmetic}$$

is more transparent.

Height function

Given $p \in \mathbb{P}^n(\mathbb{Q})$,

$$p = [x_0, \ldots, x_n]$$

with $x_0, \ldots, x_n \in \mathbb{Z}$, $\gcd(x_0, \ldots, x_n) = 1$.

Define the *height function*

$$H(p) = \max_i |x_i|.$$

Every rational embedding

$$\iota : X \to \mathbb{P}^n$$

gives a height function

$$H_\iota(x) = H(\iota(x)), \quad x \in X(\mathbb{Q}).$$

Refined basic questions

We may ask quantitative questions about

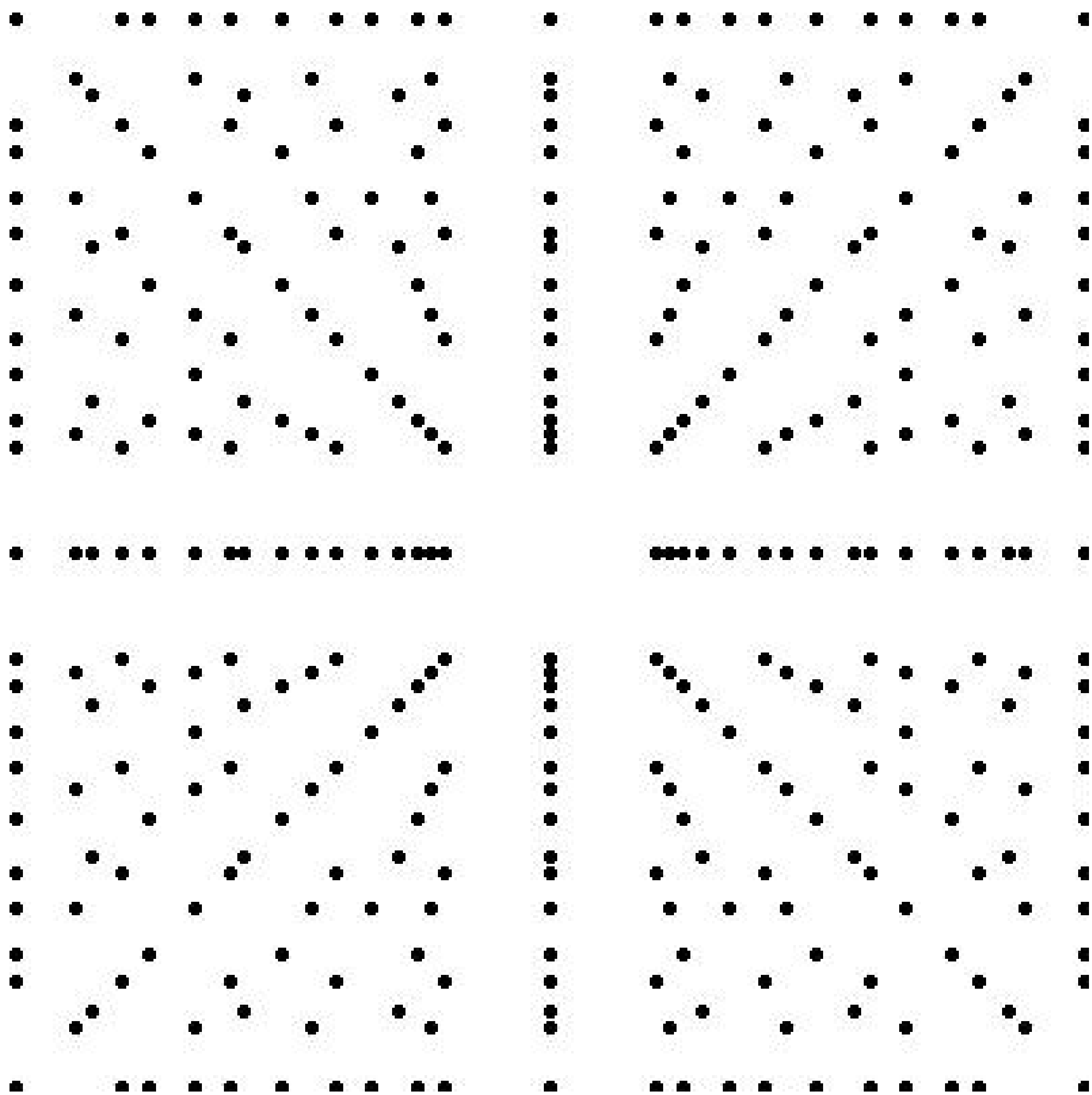
- size of $X(\mathbb{Q})$,
- distribution of $X(\mathbb{Q})$ in $X(\mathbb{R})$.

Question (size of $X(\mathbb{Q})$)
What is the order of growth of

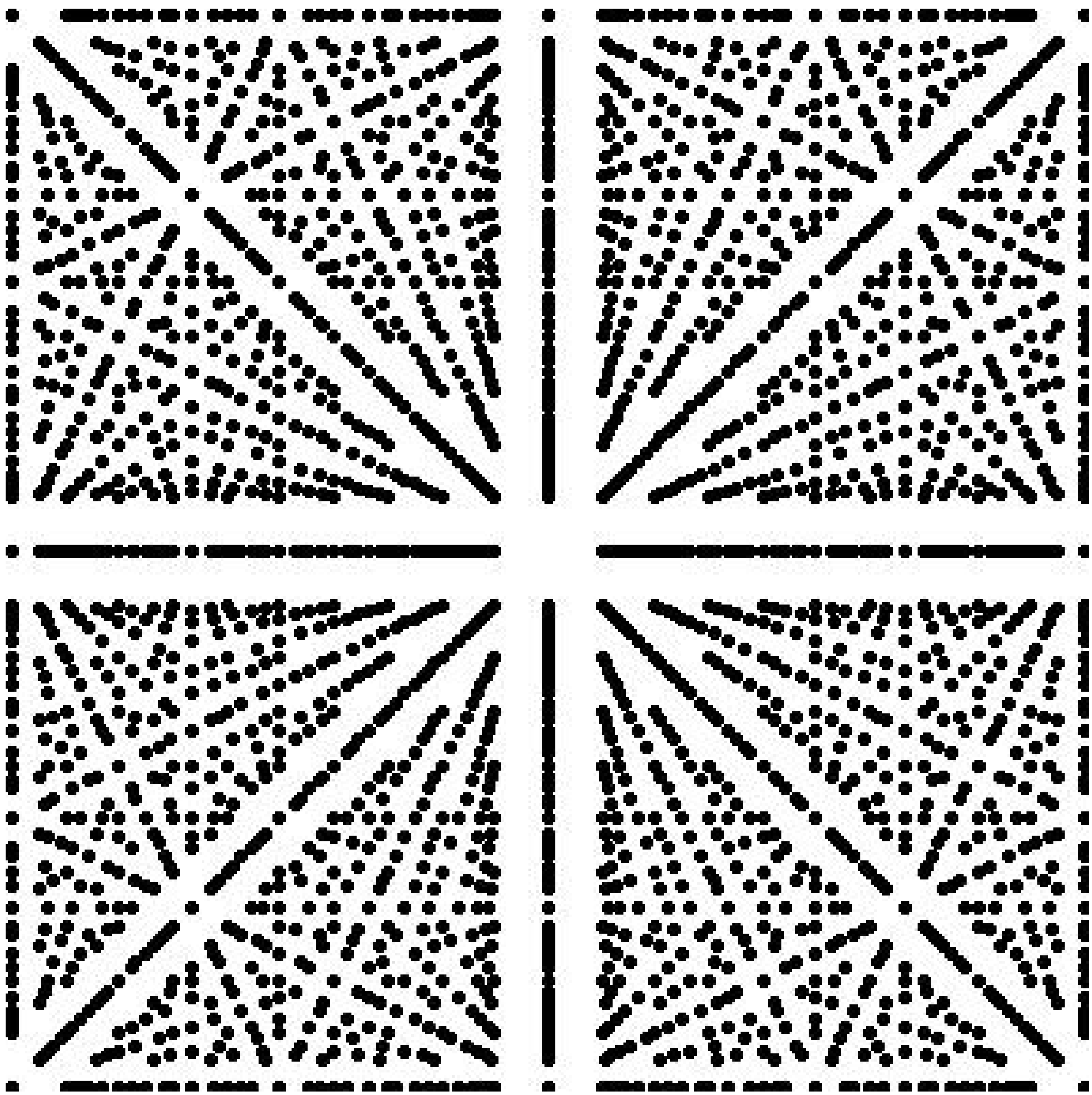
$$N_T(X,\iota) = \#\{x \in X(\mathbb{Q}) : H_\iota(x) < T\}$$

as $T \to \infty$?

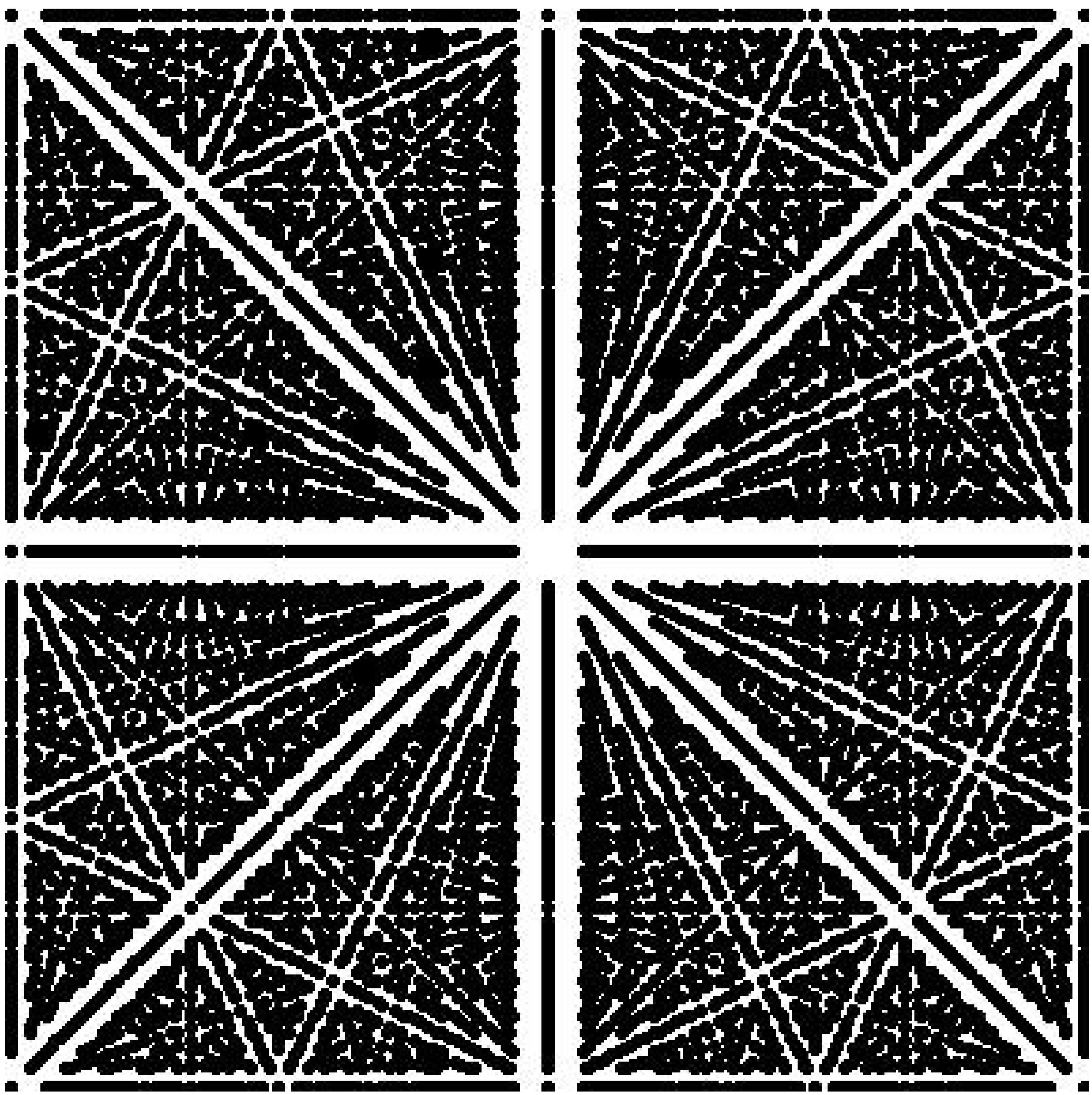
Projective Plane



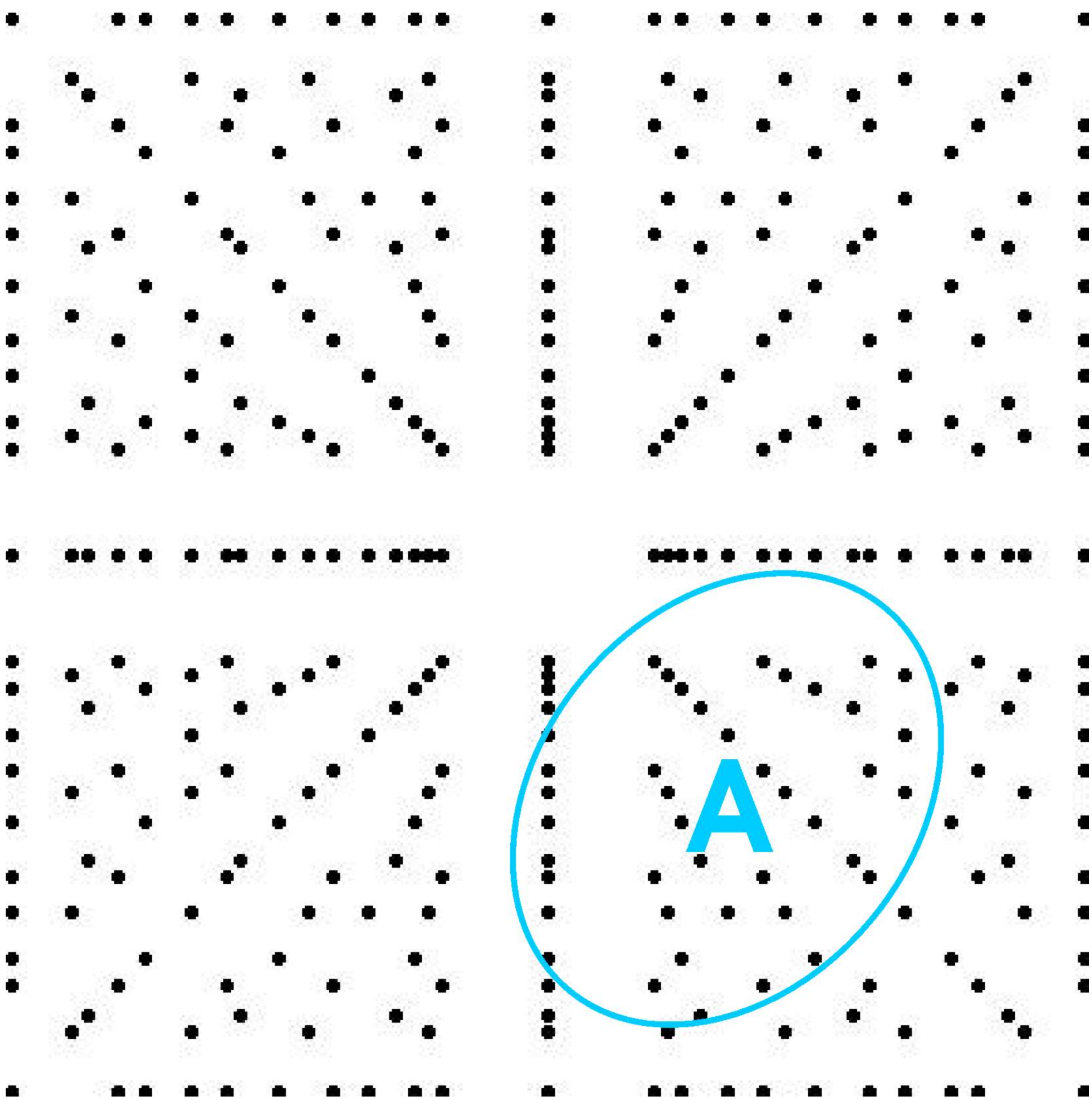
Projective Plane



Projective Plane



Projective Plane



Refined basic questions

$$\frac{\#\{x \in X(\mathbb{Q}) \cap A : \ H_\iota(x) < T\}}{\#\{x \in X(\mathbb{Q}) : \ H_\iota(x) < T\}} =?$$

Consider the probability measure on $X(\mathbb{R})$:

$$\mu_{\iota,T} = \frac{1}{N_T(X,\iota)} \sum_{x \in X(\mathbb{Q}) : H_\iota(x) < T} \delta_x .$$

Question (distribution of $X(\mathbb{Q})$)
What are the limits of $\mu_{\iota,T}$ as $T \to \infty$?

Projective Space

1. *(counting)*

$$N_T(\mathbb{P}^d) = \#\{x \in \mathbb{Z}^{d+1} : H(x) < T, \text{gcd}(x) = 1\} \sim c \cdot T^{d+1}.$$

2. *(distribution)* We make the identification

$$\mathbb{P}^d(\mathbb{R}) \simeq S^d/\{\pm 1\}.$$

Then

$$\mu_T \to c \cdot \frac{d\Omega(x)}{(\max_i |x_i|)^{d+1}} \quad \text{as } T \to \infty,$$

where $d\Omega$ is the Lebesgue measure on $S^d/\{\pm 1\}$.

Geometry of $X(\mathbb{C}) \longrightarrow X(\mathbb{Q})$

Obvious obstructions:

1. For $X = \{x^2 + y^2 + z^2 = 0\}$, $\boxed{X(\mathbb{Q}) = \emptyset.}$

   On the other hand, $X(\mathbb{Q}(i))$ is large.

   Have to pass to a finite extension of $\mathbb{Q}$!

2. Let

   $$X = \{(x_0, x_1, x_2, y_0, y_1) \in \mathbb{P}^2 \times \mathbb{P}^1 : x_0 y_1 = x_1 y_0\},$$
   $$H_{k,l}(x, y) = H(x)^k H(y)^l, \quad k, l \in \mathbb{N}.$$

   Consider

   $$Y = X \cap \{x_0 = x_1 = 0\} \simeq \mathbb{P}^1,$$
   $$X - Y \simeq \mathbb{P}^2 - \{\text{point}\}.$$

   For $k > l$,

   $$\boxed{N_T(X - Y) = o(N_T(Y))}$$

   as $T \to \infty$.

   Have to exclude "abnormal" subvarieties!

Dimension one (smooth curves)

Complete answer is known:

everything is determined by the genus g of $X(\mathbb{C})$.

$g = 0$	$g = 1$	$g \geq 2$
projective line	elliptic curves	$\{x^n + y^n = z^n\}$, $n \geq 4$
$N_T(X) \sim c \cdot T^a$	$N_T(X) \sim c \cdot (\log T)^a$	$N_T(X) < \infty$

Higher dimensions

For simplicity, we consider a smooth variety

$$X = \{f_1(x) = \cdots = f_k(x) = 0\} \subset \mathbb{P}^d$$

and assume that $\{f_i = 0\}$'s are "transversal".

Let

$\kappa = d + 1 - d_1 - \cdots - d_k$ where $d_i = \deg(f_i)$.

Heuristic argument $\Rightarrow$ $N_T(X) \approx T^{\kappa}$.

$\kappa > 0$	$\kappa = 0$	$\kappa < 0$
Fano varieties	intermediate type	general type
$X(\mathbb{Q})$ is large	?	$X(\mathbb{Q})$ is small

Fano varieties

Conjecture (Batyrev,Manin,Tschinkel)
For some open $U \subset X$,

$$N_T(U,\iota) \sim c \cdot T^{a_\iota} (\log T)^{b_\iota - 1} \quad \textit{as } T \to \infty$$

where $a_\iota \in \mathbb{Q}^+$ and $b_\iota \in \mathbb{N}$ are given explicitly in terms of the geometric invariants of X.

Conjecture (Peyre)
The measure

$$\mu_{\iota,T} = \frac{1}{N_T(X,\iota)} \sum_{x \in X(\mathbb{Q}) : H_\iota(x) < T} \delta_x$$

converges as $T \to \infty$ to an explicit smooth measure on $X(\mathbb{R})$.

Known results

These conjectures has been extensively studied for a number of homogeneous varieties:

1. flag varieties
   (Franke, Manin, Tschinkel; 1989),
2. compactifications of multiplicative groups
   (Batyrev, Tschinkel; 1995),
3. compactifications of additive groups
   (Chambert-Loir, Tschinkel; 2002),
4. compactifications of Heisenberg group
   (Shalika, Tschinkel; 2002),

We have proved the conjectures for

- the wonderful compactifications of semi-simple groups,
- the wonderful compactifications of some symmetric varieties.

Wonderful compactification

Let

$G = \mathsf{PGL}_{n+1}$,
$r : G \to \mathsf{GL}_N$ "generic" irreducible representation,
$X =$ Zarski closure of $r(G)$ in $\mathbb{P}^{N^2-1}$.

X is the *wonderful compactification* of G (de Concini–Procesi).

1. X is smooth.

2. X is a union of finitely many G-orbits.

3. G is Zariski open in X, and

$$X - G = X_1 \cup \cdots \cup X_n$$

   where X_i are smooth subvarieties intersecting transversally.

$\mathbb{Q}$-points on C.–P. compactification

Theorem (Maucourant, Oh, G.)
For every rational embedding $\iota : X \to \mathbb{P}^d$,

$$N_T(G,\iota) \sim c \cdot T^{a_\iota} (\log T)^{b_\iota - 1} \quad \text{as } T \to \infty.$$

A different proof of this theorem was also given by Shalika, Takloo-Bighash, and Tschinkel.

Theorem (Maucourant, Oh, G.)

$$\mu_{\iota,T} \to \mu_{\iota,\infty} \quad \text{as } T \to \infty$$

where $\mu_{\iota,\infty}$ is an explicit measure on $G(\mathbb{R})$.

For example, when $\iota = id$, we have

$$r(G(\mathbb{R})) \subset \mathrm{GL}_N(\mathbb{R})$$

and

$$d\mu_{\iota,\infty}(g) = \frac{d\mu_{\mathrm{haar}}(g)}{\|r(g)\|_\infty^{a_\iota}}, \quad g \in G(\mathbb{R}).$$

Integral points

Consider a variety

$$U = \{f_1 = \cdots = f_k = 0\}.$$

Question

What is the order of growth of

$$N_T(U) = \#\{x \in U(\mathbb{Z}) : \|x\|_\infty < T\}?$$

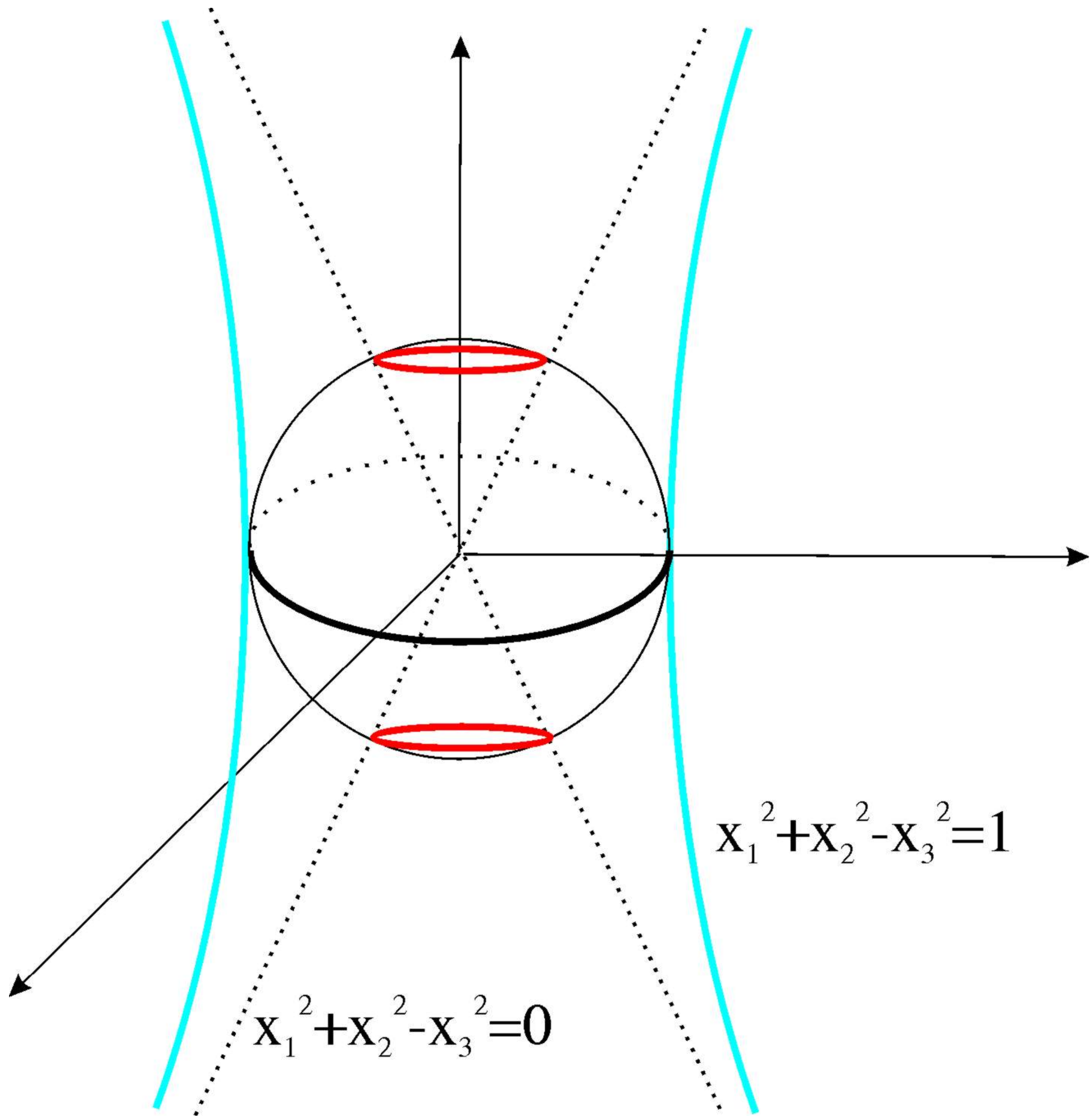
Suppose that $U \subset$ a projective variety $= X$.

Question

What are the limits of the measures

$$\nu_T = \frac{1}{N_T(U)} \sum_{x \in U(\mathbb{Z}) : \|x\|_\infty < T} \delta_x$$

in $X(\mathbb{R})$ as $T \to \infty$?



$x_1^2+x_2^2-x_3^2=1$
$x_1^2+x_2^2-x_3^2=0$

$\mathbb{Z}$-points on C.-P. compactifications

Clearly, the integral points $G(\mathbb{Z})$ are accumulating on the boundary

$$(X - G)(\mathbb{R}) = X_1(\mathbb{R}) \cup \cdots \cup X_n(\mathbb{R}).$$

Theorem (Oh, Shah, G.)
The measure ν_T converges to an explicit smooth measure supported on $\cap_{i \in I} X_i(\mathbb{R})$ for some $I \subset \{1, \ldots, n\}$.

Theorem (Oh, Shah, G.)
Let $x_0 \in (X - G)(\mathbb{R})$ and $\varepsilon > 0$ (small). Then

$$\# \left\{ x \in G(\mathbb{Z}) : \begin{array}{l} d(x, x_0) < \varepsilon \\ \|x\|_\infty < T \end{array} \right\} \sim c \cdot T^{a_{x_0}} (\log T)^{b_{x_0} - 1}.$$

The constants $a_{x_0} \in \mathbb{Q}^+$ and $b_{x_0} \in \mathbb{N}$ are determined by the location of x_0 with respect to X_i's and have explicit formulas in terms of geometric invariants of the varieties X, $X_1, \ldots, X_n$.

New results

$S =$ the space of simplectic forms
(more generally, some symmetric varieties),
$X =$ the wonderful compactification of S.

Theorem (Oh, G.)
For every rational embedding $\iota : X \to \mathbb{P}^d$,

$$N_T(S, \iota) \sim c \cdot T^{a_\iota} (\log T)^{b_\iota - 1} \quad \textit{as } T \to \infty.$$

The proof uses dynamics of unipotent flows.

Ideas behind the proof: adeles

Completions of $\mathbb{Q}$*:* $\mathbb{Q}_\infty = \mathbb{R}$, $\mathbb{Q}_p$, p − prime.

The *adele* ring is defined by

$\mathbb{A} = \{(x_p)_{p\le\infty} : x_p \in \mathbb{Q}_p,\ |x_p|_p \le 1$ for almost all $p\}$.

Then $G(\mathbb{A})$ is a *locally compact* group.

The set of rational points

$$\boxed{G(\mathbb{Q}) \hookrightarrow G(\mathbb{A}),}$$

embedded diagonally $g \mapsto (g, g, \dots)$,
is a *discrete* subgroup of $G(\mathbb{A})$ with

$$\mathrm{Vol}(G(\mathbb{Q})\backslash G(\mathbb{A})) < \infty.$$

Now the set of $\mathbb{Q}$-points can be studied using

- Harmonic analysis on $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$,
- Dynamics of subgroup actions on $G(\mathbb{Q})\backslash G(\mathbb{A})$.

Volume heuristic

One can define local height function

$$H_{\iota,p} : G(\mathbb{Q}_p) \to \mathbb{R}$$

so that

$$H_\iota(g) = \prod_p H_{\iota,p}(g) \quad \text{for } g \in G(\mathbb{Q}).$$

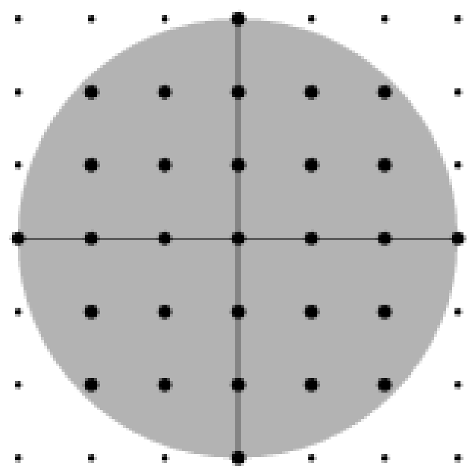
Setting

$$B_T = \{g \in G(\mathbb{A}) : H_\iota(g) < T\},$$

the question becomes:

$$\#(G(\mathbb{Q}) \cap B_T) \sim ?$$

Volume heuristic



One expects that

$$\#(G(\mathbb{Q}) \cap B_T) \sim \mathsf{Vol}(B_T)$$

when $\mathsf{Vol}(G(\mathbb{Q})\backslash G(\mathbb{A})) = 1$.

This *volume heuristic fails*, but luckily,

$$\#(G(\mathbb{Q}) \cap B_T) \sim \mathsf{Vol}(G_0 \cap B_T)$$

where G_0 is a finite index subgroup of $G(\mathbb{A})$.

Sketch of the proof

Let

$$F_T(h_1, h_2) = \sum_{\gamma \in G(\mathbb{Q})} \chi_{B_T}(h_1^{-1}\gamma h_2).$$

We have to show that

$$F_T(e, e) \sim \mathrm{Vol}(B_T).$$

For a "bump" function

$$\alpha(h_1, h_2) = \alpha_1(h_1)\alpha_2(h_2), \quad h_1, h_2 \in G(\mathbb{Q})\backslash G(\mathbb{A}),$$

we have, after a *change of variable*,

$$\langle F_T, \alpha \rangle = \cdots = \int_{B_T} \langle g \cdot \alpha_1, \alpha_2 \rangle dg.$$

If $g \cdot \alpha_1$ and α_2 *become independent* as $g \to \infty$:

$$\boxed{\langle g \cdot \alpha_1, \alpha_2 \rangle \to (\textstyle\int \alpha_1) \cdot (\textstyle\int \alpha_2),} \qquad \text{(*)}$$

then

$$\begin{aligned} \langle F_T, \alpha \rangle &= \int_{B_T} \langle g \cdot \alpha_1, \alpha_2 \rangle dg \\ &\sim \mathsf{Vol}(B_T) \cdot \left(\int \alpha_1 \right) \cdot \left(\int \alpha_2 \right), \end{aligned}$$

i.e.,

$$F_T \sim \mathsf{Vol}(B_T) \qquad \text{weakly}$$

and, in fact, pointwise too.

(*) is deduced from *quantitative property* τ.

Property τ

Theorem (Clozel, 2003)
Let π be an irreducible representation

$$\pi = \otimes_p \pi_p \subset L^2(G(\mathbb{Q})\backslash G(\mathbb{A})).$$

Then $\pi_p \neq 1$ is uniformly isolated from 1 in $\widehat{G(\mathbb{Q}_p)}$.

This theorem is the culmination of the work of many people:

- Jacquet–Langlands,
- Gelbart–Jacquet,
- Rogawski,
- Burger-Sarnak,
- Clozel.

Quantitative property τ

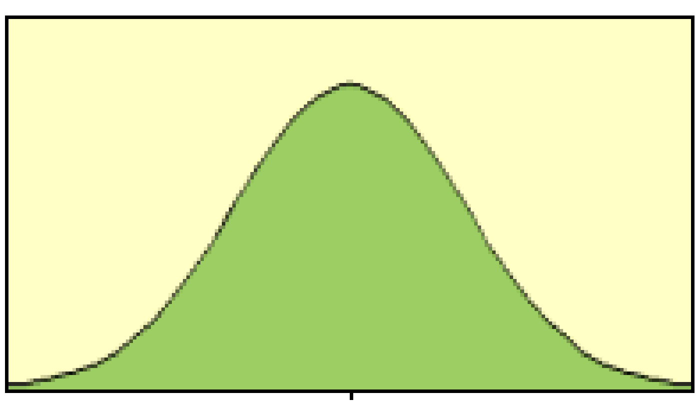
$L^2(G(\mathbb{Q})\backslash G(\mathbb{A})) = L^2_{00} \perp \{$1-dim. representations$\}$.

Theorem (Maucourant, Oh, G.)
There exists $\delta > 0$ such that for any smooth vectors

$$v = \otimes_p v_p,\ w = \otimes_p w_p \in L^2_{00},$$

we have

$$|\langle g \cdot v_p, w_p \rangle| \le c(v_p, w_p) H_{\iota,p}(g)^{-\delta}, \quad g \in G(\mathbb{Q}_p).$$

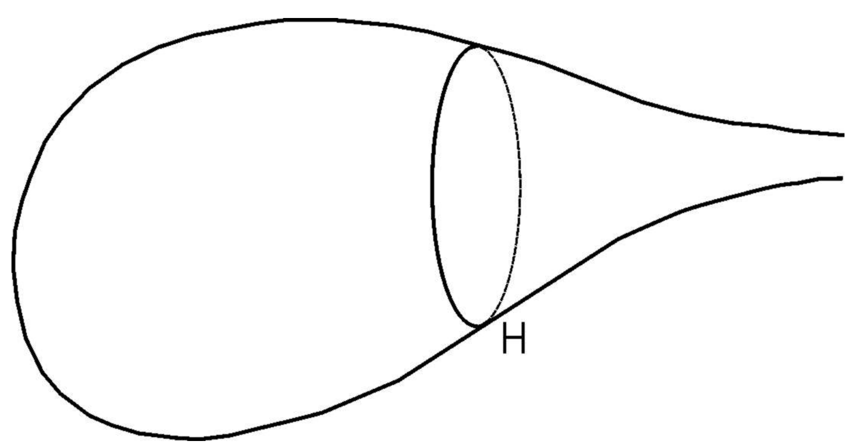


Dynamical approach: motivation

$M =$ hyperbolic surface of finite area,
$T^1M =$ unit tangent bundle of $M \simeq \Gamma\backslash \mathrm{PSL}(2,\mathbb{R})$,
$h_t : T^1M \to T^1M$ — horocyclic flow

$$x \mapsto x \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

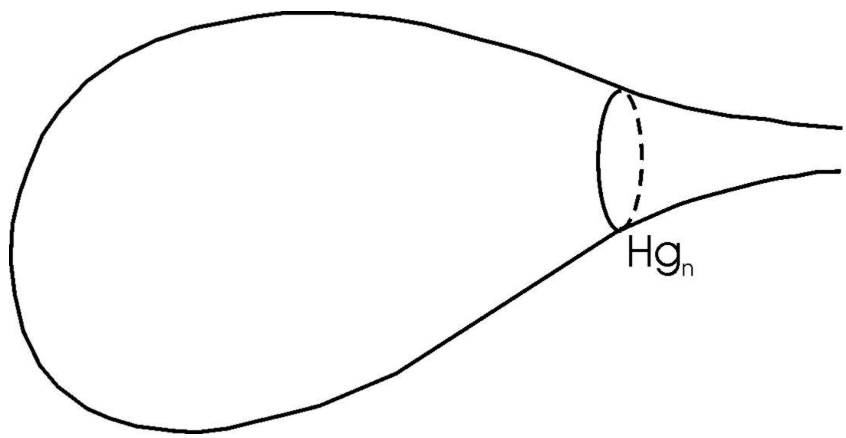
Let $\mathcal{H} \subset T^1M$ be a closed orbit of h_t.



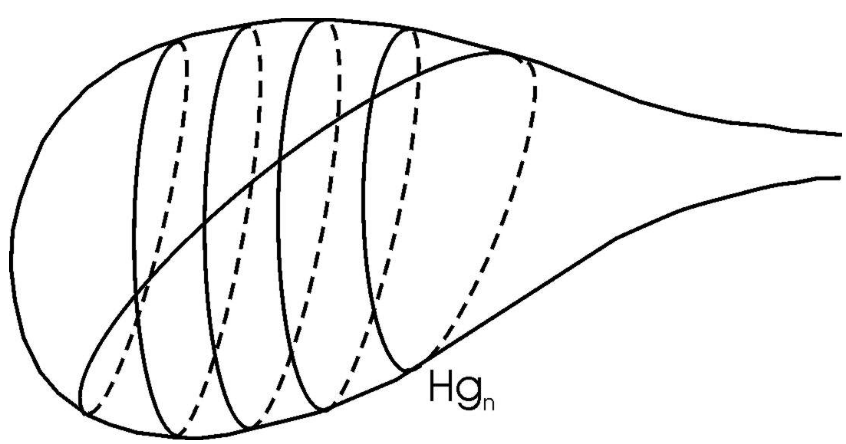
For a sequence of closed orbits

$$\mathcal{H}g_n \quad \text{with } g_n \to \infty \text{ in } \{h_t\}\backslash \mathrm{PSL}(2,\mathbb{R}),$$

- $\mathcal{H}g_n \to \infty$,

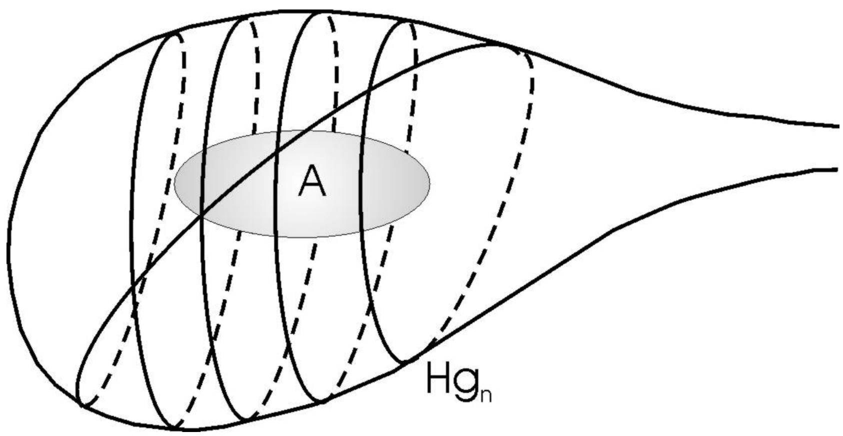


- $\mathcal{H}g_n$ is asymptotically dense.



Equidistribution

In the second case, the sequence $\mathcal{H}g_n$ becomes equidistributed:



$$\frac{\ell(\mathcal{H}g_n \cap A)}{\ell(\mathcal{H}g_n)} \to \text{Area}(A) \qquad \text{as } n \to \infty.$$

To count rational points,
we need similar equidistribution result for the space $G(\mathbb{Q})\backslash G(\mathbb{A})$.

Dynamical approach

Fix $v_0 \in X(\mathbb{Q})$.

Group G acts on X and $L = \mathrm{Stab}_G(v_0)$.

Aim: *compute the asymptotics of*

$$\#(v_0 G(\mathbb{Q}) \cap B_T) = \{x \in v_0 G(\mathbb{Q}) : H_\iota(x) < T\}.$$

Assume that $\mathcal{L} = G(\mathbb{Q})L(\mathbb{A})$ has finite volume.

Let

$$F_T(g) = \sum_{v \in v_0 G(\mathbb{Q})} \chi_{B_T}(vg).$$

For $\alpha : G(\mathbb{Q})\backslash G(\mathbb{A}) \to \mathbb{R}$ with compact support,

$$\langle F_T, \alpha \rangle = \cdots = \int_{B_T} \left(\int \alpha \, d\nu_g \right) dg$$

where ν_g is the measure supported on $\mathcal{L}g$.

Equidistribution

Theorem (Oh, G.)
Let

$$G = \mathrm{PGL}_{2n},$$
$$L = \mathrm{PSp}_{2n},$$
$$\nu_g = \text{the measure supported on } \mathcal{L}g = G(\mathbb{Q})L(\mathbb{A})g \subset G(\mathbb{Q})\backslash G(\mathbb{A}).$$

Then for any continuous $\alpha : G(\mathbb{Q})\backslash G(\mathbb{A}) \to \mathbb{R}$ *with compact support,*

$$\boxed{\int \alpha \, d\nu_g \to \int \alpha \, d\mu \quad \text{as} \quad g \to \infty \quad \text{in} \quad L(\mathbb{A})\backslash G(\mathbb{A})}$$

where

$$\mu = \begin{array}{l} \text{measure invariant under} \\ \text{a finite index of subgroup of } G(\mathbb{A}). \end{array}$$

The proof uses Ratner theory of unipotent flows.