Homework problem set

(1) Let (x_n) be a sequence of real numbers such that
$$x_{m+n} \le x_m + x_n.$$
Then the following limit exists
$$\lim_{n\to\infty} \frac{x_n}{n} = \lim_{n\ge 1} \frac{x_n}{n}.$$

(2) Let μ be a probability measure on $\mathrm{SL}_2(\mathbb{R})$ defined by
$$\mu = p_1\delta_{a_1} + p_2\delta_{a_2},$$
where $p_1, p_2 > 0$, $p_1 + p_2 = 1$ and
   (i) $a_1 = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$, $a_2 = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix}$ with $\lambda > 1$,
   (ii) $a_1 = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$, $a_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ with $\lambda > 1$.

   Compute the top Lyapunov exponent of the random products defined by μ.

(3) Let ν be a probability measure on $\mathrm{GL}(d, \mathbb{C})$, and $\Omega = \mathrm{GL}(d, \mathbb{C})^{\mathbb{N}}$ equipped with the product measure $\mathbb{P} = \nu^{\otimes \mathbb{N}}$.
   (i) Show that the shift-map
$$\theta : \Omega \to \Omega : (\omega_n) \mapsto (\omega_{n+1})$$
   is mixing, that is, for every measurable $A, B \subset \Omega$,
$$\mathbb{P}(\theta^{-n}A \cap B) \to \mathbb{P}(A)\mathbb{P}(A) \quad \text{as } n \to \infty.$$
   (ii) Show that every measurable θ-invariant subset of Ω has measure zero or one.

(4) With notation as in Lecture 2, verify that
   (i) $\|\pi(g)f\|_2 = \|f\|_2$ for all $g \in \mathrm{SL}(d, \mathbb{C})$ and $f \in \mathcal{H}$,
   (ii) $\|P_\nu f\|_2 \le \|f\|_2$ for all $f \in \mathcal{H}$,
   (iii) $P_{\nu_1} P_{\nu_2} = P_{\nu_1 * \nu_2}$,
   (iv) $P_\nu^* = P_{\nu^\vee}$ where $d\nu^\vee(g) = d\nu(g^{-1})$.

(5) Let A_n be a sequence of matrices in $\mathrm{GL}(d, \mathbb{C})$. Prove that the following statements are equivalent:
   (i) For some proper measure ν on $\mathbb{P}(\mathbb{C}^d)$, $A_n\nu$ converges to a Dirac measure δ_z.
   (ii) For any proper measure ν on $\mathbb{P}(\mathbb{C}^d)$, $A_n\nu$ converges to a Dirac measure δ_z.
   (iii) All limit points of $\frac{A_n}{\|A_n\|}$ are rank-one matrices with the range $\langle z \rangle$.

(6) Prove that a closed subgroup of $\mathrm{GL}(d, \mathbb{C})$ acts totally irreducibly on $\mathbb{C}^d$ if and only if each closed normal subgroup of finite index acts irreducibly.

(7) Let ν be a proper probability measure on $\mathbb{P}(\mathbb{C}^d)$. Let

$$G = \{g \in \mathrm{GL}(d, \mathbb{C}) : g \cdot \nu = \nu\}.$$

Prove that every limit point of the sequence $\frac{g_n}{\|g_n\|}$ with $g_n \in G$ is nondegenerate.

(8) With notation as in Lecture 4 (page 4), show that

$$E(\phi, \mathcal{F}_n) = \int_\Omega \phi(\omega) \prod\nolimits_{i \geq n+1} d\mu(\omega_i).$$

(9) Show that if $\{\phi_n\}$ is a martingale with respect to σ-algebras $\mathcal{F}_n$, then
   (i) $E(\phi_{n+k}|\mathcal{F}_n) = \phi_n$ for all $k \geq 0$,
   (ii) $E(\phi_{n+1}) = E(\phi_n)$.

(10) Show that a sequence of real numbers (x_n) converges (possibly to $\pm\infty$) iff $u_\infty(a, b) < \infty$ for all intervals (a, b) iff $u_\infty(a, b) < \infty$ for all intervals (a, b) with $a, b \in \mathbb{Q}$.

(11) Let μ a probability measure on $\mathrm{GL}(d, \mathbb{C})$ and assume that its Lyapunov exponents satisfy

$$\lambda_1 > \cdots > \lambda_d.$$

Show that the action of the semigroup S_μ, the closed semigroup generated by the support of μ, is contracting on $\wedge^i \mathbb{C}^d$ for $i = 1, \ldots, d-1$.

(12) Consider a sequence of independent identically distributed matrices $X_n \in \mathrm{GL}(d, \mathbb{C})$, $n \geq 1$, such that

$$X_n = \begin{pmatrix} A_n & B_n \\ 0 & C_n \end{pmatrix}.$$

Let $\lambda, \lambda_1, \lambda_2$ denote the top Lyapunov exponents of X_n, A_n, B_n respectively. Prove that

$$\lambda = \max\{\lambda_1, \lambda_2\}.$$

(13) Suppose that $S \subset \mathrm{GL}(d, \mathbb{C})$ is contacting on $\wedge^i \mathbb{C}^d$. Prove that the set $\{s^{-1} : s \in S\}$ is contacting on $\wedge^{d-i} \mathbb{C}^d$.