Lecture 1: Lyapunov exponents and subadditive ergodic theorem.

Introduction.

Let ν be a probability measure on $GL(d, \mathbb{C})$, and $X_1, \dots X_n, \dots$ be independent random matricies with distribution ν. That is,

for $\mathcal{U}_1, \dots \mathcal{U}_s \subset GL(d, \mathbb{C})$,

$$\mathrm{Prob}(X_{i_1} \in \mathcal{U}_1, \dots, X_{i_s} \in \mathcal{U}_s) = \nu(\Omega_1) \dots \nu(\Omega_s).$$

<u>Question</u>: Determine the asymptotic behaviour of the product $S_n = X_n \dots X_1$ as $n \to \infty$

<u>Rmk.</u> This problem is analogous to the classical <u>law of large numbers</u> and <u>central limit theorem</u>.

Let $\Omega = GL(d, \mathbb{C})^{\mathbb{N}}$, equipped with the product measure $\mathbb{P} = \nu^{\otimes \mathbb{N}}$,

and $\theta : \Omega \to \Omega : (\omega_n) \longmapsto (\omega_{n+1})$

be the shift map.

Setting $X : \Omega \to GL(d, \mathbb{C}) : (\omega_n) \mapsto \omega_1$, we have

$$S_n(\omega) = X(\theta^{n-1}\omega)\, X(\theta^{n-2}\omega) \dots X(\omega), \quad \omega \in \Omega.$$

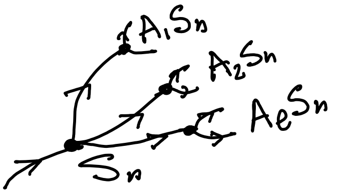
More generally, take a prob. space $(\Omega, \mathbb{P})$ with a measure-preserving transformation $\theta : \Omega \to \Omega$ and a map $X : \Omega \to GL(d, \mathbb{C})$.

<u>Question</u>: What is the asymptotic behaviour of

$$S_n(\omega) = X(\theta^{n-1}\omega) \dots X(\omega) \, ?$$

<u>Applications</u>

1) Random walks: Fix matrices $A_1, \dots A_\ell \in GL(d, \mathbb{C})$,



and generate a sequence $S_0 = I$, $S_{n+1} = A_i \cdot S_n$ with probability $\frac{1}{\ell}$.

2) Smooth dynamical systems

$F : \mathcal{U} \to \mathcal{U}$ $(\mathcal{U} \subset \mathbb{R}^d)$ – smooth map.

By the chain rule, $D(F^n)_u = (DF)_{F^{n-1}u} \dots (DF)_u$.

The growth rate of $DF^n)_u$ is related to entropy of F and dimension of invariant measures.

3) Random Schrodinger operators.

$$\mathcal{H} = \ell^2(\mathbb{Z}) = \{ (a_n)_{n\in\mathbb{Z}} : \sum_{n\in\mathbb{Z}} |a_n|^2 < \infty \}.$$

$V: \Omega \to \mathbb{R}$ (potential)

$$H: \mathcal{H} \to \mathcal{H} : (a_n) \longmapsto \left(a_{n-1} + a_{n+1} + V(\theta^n \omega) a_n \right)$$

Consider the eigenvalue equation:

$$Ha = Ea \quad \text{for } E \in \mathbb{R}.$$

$$\Updownarrow$$

$$a_{n+1} + a_{n-1} + V(\theta^n \omega) a_n = E a_n$$

The solutions are determined by recurrence relation:

$$\begin{pmatrix} a_{n+1} \\ a_n \end{pmatrix} = \begin{pmatrix} E - V(\theta^n \omega) & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_n \\ a_{n-1} \end{pmatrix}.$$

This again leads to products of random matrices...

Lyapunov Exponents.

$(\Omega, \mathbb{P}, \theta)$ - prob. space with measure-preserving transformation.

$X: \Omega \to GL(d, \mathbb{C})$.

Assume that $\int_\Omega \log^+ \|X(\omega)\| \, d\omega < \infty$.

$$S_n(\omega) = X(\theta^{n-1}\omega) \ldots X(\omega).$$

Thm (Furnstenberg–Kesten)

The limit $\lambda(\omega) \overset{def}{=} \lim_{n\to\infty} \frac{1}{n} \log \|S_n(\omega)\|$ exists for a.e. $\omega \in \Omega$.

The function λ is θ-inv. and satisfies

$$\int_\Omega \lambda(\omega)\, d\omega = \lim_{n\to\infty} \frac{1}{n} \int_\Omega \log \|S_n(\omega)\|\, d\omega.$$

Def. $\lambda(\omega)$ is the top Lyapunov exponent.

Let $a_n = \int_\Omega \log \|S_n(\omega)\|\, d\omega$. We have

$$\begin{aligned} a_{n+m} &= \int_\Omega \log \| X(\theta^{n+m-1}\omega) \dots X(\omega)\|\, d\omega \\ &\le \int_\Omega \log \|X(\theta^{n+m-1}\omega) \dots X(\theta^m\omega)\|\, d\omega + \int_\Omega \log \|X(\theta^{m-1}\omega) \dots X(\omega)\|\, d\omega \\ &= \int_\Omega \log \|S_n(\omega)\|\, d\omega + \int_\Omega \log \|S_m(\omega)\|\, d\omega \\ &= a_n + a_m. \end{aligned}$$

Exercise: If a sequence $\{a_n\}$ satisfies $a_{n+m} \le a_n + a_m$, then $\lim_{n\to\infty} \frac{a_n}{n} = \inf_{n\ge 1} \frac{a_n}{n}$.

This implies, in particular, that $\lim_{n\to\infty} \frac{1}{n} \int_\Omega \log \|S_n(\omega)\|\, d\omega$ exists.

Let $\varphi_n(w) = \log \| S_n(w) \|$.

As above, we check that

$$\boxed{\varphi_{n+m}(w) \le \varphi_n(\Theta^m w) + \varphi_m(w)}$$

We call such sequences of functions subadditive.

Furstenberg-Kesten Thm would follow from

Subadditive Ergodic Thm (Kingman)

Let $\varphi_n : \Omega \to \mathbb{R}$ be a subadditive sequence with $\varphi_1^+ \in L^1$.

Then $\varphi(w) = \lim\limits_{n\to\infty} \dfrac{\varphi_n(w)}{n}$ exists a.e., and

$$\int_\Omega \varphi(w)\, dw = \lim_{n\to\infty} \frac{1}{n} \int_\Omega \varphi_n(w)\, dw \overset{def}{=} L.$$

Let $\varphi_-(w) = \varliminf\limits_{n\to\infty} \dfrac{\varphi_n(w)}{n}$ and $\varphi_+(w) = \varlimsup\limits_{n\to\infty} \dfrac{\varphi_n(w)}{n}$.

Lem. 1. $\varphi_- \circ \Theta \overset{a.e.}{=} \varphi_-$ and $\varphi_+ \circ \Theta \overset{a.e.}{=} \varphi_+$.

Proof. By subadditivity, $\varphi_-(w) \le \varliminf\limits_{n\to\infty} \dfrac{\varphi_n(\Theta w) + \varphi_1(w)}{n+1}$

$$= \varphi_-(\Theta w).$$

Hence, $\Theta^{-1}(\{\varphi_- \ge a\}) \supset \{\varphi_- \ge a\}$ for all $a \in \mathbb{R}$.

Since Θ is measure-preserving,

$$\Theta^{-1}(\{\varphi_- \ge a\}) \overset{a.e.}{=} \{\varphi_- \ge a\}$$

This implies that $\varphi_- \circ \Theta = \varphi_-$ a.e.

Fix $\varepsilon > 0$. For $k \in \mathbb{N}$, define

$$E_k = \left\{ \omega \in \Omega : \frac{\varphi_j(\omega)}{j} \le \varphi_-(\omega) + \varepsilon \text{ for some } j = \overline{1,k} \right\}.$$

Note that $E_k \subset E_{k+1}$ and $\bigcup_k E_k = \Omega$.

Let $$\psi_k(\omega) = \begin{cases} \varphi_-(\omega) + \varepsilon, & \omega \in E_k, \\ \varphi_1(\omega), & \omega \in E_k^c. \end{cases}$$

<u>Lem. 2</u> $\forall n > k$ and a.e. $\omega \in \Omega$:

$$\varphi_n(\omega) \le \sum_{i=0}^{n-k-1} \psi_k(\theta^i \omega) + \sum_{i=n-k}^{n-1} \max\{\psi_k, \varphi_1\}(\theta^i \omega).$$

<u>Proof</u> By Lem. 1, we may assume that $\varphi_-(\theta^n \omega) = \varphi_-(\omega)$ for all n.

Define a sequence $m_0 \le n_1 < m_1 \le n_2 < \dots$

Let $m_0 = 0$, n_j = least integer $\ge m_{j-1}$ such that $\theta^{n_j}\omega \in E_k$ (if it exists).

Then by definition of E_k, there exists m_j such that

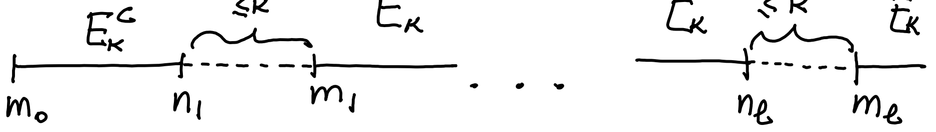
$$1 \le m_j - n_j \le k, \quad \boxed{\varphi_{m_j - n_j}(\theta^{n_j}\omega) \le (m_j - n_j)\left(\varphi_-(\theta^{n_j}\omega) + \varepsilon\right).} \quad (*)$$

Now let l be the largest integer such that $m_l \le n$.

By subadditivity,

$$\boxed{\varphi_n(\omega) \le \sum_{i \in I} \varphi_1(\theta^i \omega) + \sum_{j=1}^{l} \varphi_{m_j - n_j}(\theta^{n_j}\omega)} \quad (**)$$

where $I = \bigcup_{j=0}^{l-1} [m_j, n_{j+1}) \cup [m_l, n)$.



For $i \in \bigcup_{j=0}^{\ell-1} [m_j, n_{j+1}) \cup [m_\ell, \min\{n_{\ell+1}, n\})$, we have

$\varphi_1(\theta^i w) = \psi_k(\theta^i w)$ because $\theta^i w \in E_k^c$.

Since $\varphi_-(\theta^n w) = \varphi_-(w)$ and $\psi_k \geq \varphi_- + \varepsilon$, by $(*)$

$$\varphi_{m_j - n_j}(\theta^{n_j} w) \leq \sum_{i=n_j}^{m_j-1} (\varphi_-(\theta^i w) + \varepsilon) \leq \sum_{i=n_j}^{m_j-1} \psi_k(\theta^i w).$$

Finally, the estimate follows from $(**)$.

Lem. 3. $\int_\Omega \varphi_-(w)\, dw = L$.

Proof. First, suppose that $\frac{\varphi_n}{n} \geq -C$ uniformly.

Then by Fatou Lemma, $\int_\Omega \varphi_-(w)\, dw \leq \lim_{n\to\infty} \int_\Omega \frac{\varphi_n(w)}{n} dw = L$.

To prove the opposite inequality, we observe that by Lem. 2,

$$\frac{1}{n} \int_\Omega \varphi_n(w)\, dw \leq \frac{n-k}{n} \int_\Omega \psi_k(w)\, dw + \frac{k}{n} \cdot \int_\Omega \max\{\psi_k, \varphi_1\}(w)\, dw.$$

Taking $n \to \infty$, we obtain $L \leq \int_\Omega \psi_k(w)\, dw$.

Since $\psi_k \xrightarrow[k\to\infty]{} \varphi_- + \varepsilon$, it follows that

$$L \leq \int_\Omega \varphi_-(w)\, dw + \varepsilon \quad \text{for every } \varepsilon > 0.$$

This proves the lemma under additional assumption.

In general, we set $\varphi_n^C = \max\{\varphi_n, -Cn\}$,
$$\varphi_-^C = \max\{\varphi_-, -C\}.$$

By the monotone convergence Thm,
$$\int_\Omega \varphi_-(w)\,dw = \inf_C \int_\Omega \varphi_-^C(w)\,dw = \inf_C \inf_n \int_\Omega \frac{\varphi_n^C(w)}{n}\,dw$$
$$= \inf_n \int_\Omega \frac{\varphi_n(w)}{n}\,dw = L.$$

$\blacksquare$

Lem. 4. For every $\varphi \in L^1$, $\lim\limits_{n\to\infty} \frac{\varphi(\theta^n w)}{n} = 0$ a.e.

Proof. $\sum\limits_{n=1}^{\infty} |\{|\varphi\circ\theta^n| \geq \varepsilon n\}| = \sum\limits_{n=1}^{\infty} |\{|\varphi| \geq \varepsilon n\}|$
$$= \sum_{n=1}^{\infty} \sum_{k\geq n} |\{k \leq \tfrac{|\varphi|}{\varepsilon} < k+1\}| = \sum_{k\geq 1} k\,|\{k \leq \tfrac{|\varphi|}{\varepsilon} < k+1\}|$$
$$\leq \int_\Omega \frac{|\varphi(w)|}{\varepsilon}\,dw < \infty.$$

Let $\Omega_\varepsilon = \{w : \frac{|\varphi(\theta^n w)|}{n} > \varepsilon$ for infinitely many $n\}$.
$$= \bigcap_{k\geq 1} \bigcup_{n\geq k} \{|\varphi\circ\theta^n| \geq \varepsilon n\}$$
$$|\Omega_\varepsilon| \leq \sum_{n\geq k} |\{|\varphi\circ\theta^n| \geq \varepsilon n\}| \xrightarrow[k\to\infty]{} 0.$$

Finally, $\Omega_0 = \bigcup_{i\geq 1} \Omega_{1/i}$ has measure 0, and
$\forall \omega \notin \Omega_0$: $\frac{|\varphi(\theta^n \omega)|}{n} > \varepsilon$ has only fin. many solutions.

Lem. 5. $\overline{\lim_{n\to\infty}} \frac{\varphi_{nk}}{n} \overset{a.e.}{=} k \cdot \overline{\lim_{n\to\infty}} \frac{\varphi_n}{n}$

Proof. The inequality "$\leq$" is obvious.
To prove the opposite inequality, we write $n = kq_n + r_n$ with $r_n = \overline{1,k}$. By subadditivity,

$$\varphi_n \leq \varphi_{kq_n} + \varphi_{r_n} \circ \theta^{kq_n} \leq \varphi_{kq_n} + \psi \circ \theta^{kq_n},$$

where $\psi = \max\{\varphi_1^+, \dots \varphi_k^+\}$.
Since $\psi \in L^1$, by Lem. 4, $\frac{\psi \circ \theta^{kq_n}}{q_n} \xrightarrow[n\to\infty]{a.e} 0$.

Since $\frac{n}{q_n} \to k$ as $n \to \infty$,

$$\overline{\lim_{n\to\infty}} \frac{\varphi_n}{n} \overset{a.e.}{\leq} \overline{\lim_{n\to\infty}} \frac{1}{n} \cdot \varphi_{kq_n} = \frac{1}{k} \cdot \overline{\lim_{n\to\infty}} \frac{1}{q_n} \varphi_{kq_n} \leq \frac{1}{k} \overline{\lim_{n\to\infty}} \frac{\varphi_{nk}}{n}.$$

Lem. 6. Assume that $\inf_{\omega} \varphi_n(\omega) > -\infty$.
Then $\int_{\Omega} \varphi_+(\omega)\, d\omega \leq L$.

Proof. Let $\Phi_n = -\sum_{j=0}^{n-1} \varphi_k \circ \theta^{kj}$.

Clearly, $\Phi_{n+m} = \Phi_m + \Phi_n \circ \Theta^{km}$, and $\Phi_1 = -\varphi_k \in L^1$.

Hence, by Lem. 3,

$$\int_\Omega \Phi_-(\omega)\,d\omega = \lim_{n\to\infty} \int_\Omega \frac{\Phi_n(\omega)}{n}\,d\omega \underset{\text{by invariance.}}{=} -\int_\Omega \varphi_k(\omega)\,d\omega$$

On the other hand, by subadditivity and Lemma 5,

$$-\Phi_- = \overline{\lim_{n\to\infty}}\, \frac{1}{n}\sum_{j=0}^{n-1} \varphi_k \circ \Theta^{jk} \geq \overline{\lim_{n\to\infty}}\, \frac{\varphi_{kn}}{n} = k\cdot \varphi_+ .$$

Hence, $\int_\Omega \varphi_+(\omega)\,d\omega \leq -\frac{1}{k}\int_\Omega \Phi_-(\omega)\,d\omega = \frac{1}{k}\int_\Omega \varphi_k(\omega)\,d\omega$.

Since k is arbitrary, this implies the lemma. $\square$

Let $\varphi_n^C = \max\{\varphi_n, -Cn\}$, $\varphi_-^C = \max\{\varphi_-, -C\}$, $\varphi_+^C = \max\{\varphi_+, -C\}$.

Lem. 3 & 6 imply that $\int_\Omega \varphi_-^C(\omega)\,d\omega = \int_\Omega \varphi_+^C(\omega)\,d\omega$.

Since $\varphi_-^C(\omega) \leq \varphi_+^C(\omega)$, $\varphi_-^C \overset{a.e.}{=} \varphi_+^C$.

Since, $\varphi_-^C \xrightarrow[C\to\infty]{} \varphi_-$ and $\varphi_+^C \xrightarrow[C\to\infty]{} \varphi_+$,

it follows that $\varphi_+ \overset{a.e.}{=} \varphi_-$. $\square$