Lecture II: Positivity of Lyapunov exponents.

Let ν be a probability measure on $GL(d,\mathbb{C})$, and $X_1, \dots X_n, \dots$ independent random matrices with distribution ν. We are interested in the growth rate:

$$\overline{\lim_{n\to\infty}} \frac{1}{n} \log \| X_1 \dots X_n \|.$$

Formally, we set $\Omega = GL(d,\mathbb{C})^{\mathbb{N}}$ (with measure $\mathbb{P} = \nu^{\otimes \mathbb{N}}$)

$\theta : \Omega \to \Omega : (\omega_n) \mapsto (\omega_{n+1})$, $X : \Omega \to GL(d,\mathbb{C}) : (\omega_n) \mapsto \omega_1$.

Then the limit becomes:

$$\lambda(\omega) = \overline{\lim_{n\to\infty}} \frac{1}{n} \log \| X(\theta^{n-1}\omega) \dots X(\omega) \|.$$

By Furstenberg-Kesten Thm, it exists a.e., and is θ-invariant.

Exercise. 1) Show that θ is mixing, that is, $\forall$ measurable $A, B \subset \Omega$: $\mathbb{P}(\theta^{-n} A \cap B) \xrightarrow[n\to\infty]{} \mathbb{P}(A)\mathbb{P}(B)$.

2) Deduce that every measurable θ-inv. subset of Ω has $\mathbb{P}$-measure 0 or 1.

Since λ is θ-invariant, $\lambda = \text{const}$ a.e.

$$\int_\Omega \lambda(\omega)\, d\omega = \lim_{n\to\infty} \frac{1}{n} \int_\Omega \log \| X(\theta^{n-1}\omega) \dots X(\omega) \| \, d\mathbb{P}(\omega)$$

$$= \lim_{n\to\infty} \frac{1}{n} \int_\Omega \log \| \omega_n \dots \omega_1 \| \, d\nu(\omega_n) \dots d\nu(\omega_1).$$

<u>Def.</u> For measures ν_1, ν_2 on $GL(d,\mathbb{C})$, we define a measure $\nu_1 * \nu_2$ on $GL(d,\mathbb{C})$ by

$$\int_{GL(d,\mathbb{R})} f(g)\, d(\nu_1 * \nu_2)(g) = \int_{GL(d,\mathbb{R}) \times GL(d,\mathbb{R})} f(g_1 g_2)\, d\nu_1(g_1)\, d\nu_2(g_2).$$

$$\boxed{\lambda(\omega) \overset{a.e.}{=} \lim_{n\to\infty} \frac{1}{n} \int_{\Omega} \log \|g\|\, d\nu^{*n}(g)}$$

When $\lambda(\omega) > 0$?

<u>Notation:</u> · $supp(\nu)$ = smallest closed subset F with $\nu(F)=1$
· G_ν = closed subgroup generated by $supp(\nu)$.

<u>Def</u> $G < GL(d,\mathbb{C})$ is <u>totally irreducible</u> if there is no finite union of proper subspaces of $\mathbb{C}^d$ which is G-invariant.

For simplicity, assume that $G_\nu \subset SL(d,\mathbb{C})$.

<u>Thm</u> (Furstenberg) Assume that G_ν is totally irreducible and noncompact. Then

$$\boxed{\lambda(\omega) \overset{a.e.}{>} 0.}$$

Let $\mathcal{H} = L^2(\mathbb{C}^d) = \{ f: \mathbb{C}^d \to \mathbb{C} : \int |f|^2 < \infty \}$.

For $g \in SL(d,\mathbb{C})$, $\pi(g): \mathcal{H} \to \mathcal{H} : f \to f(g^{-1}x)$.

We also define $P_\nu : \mathcal{H} \to \mathcal{H} : f \longmapsto \int_{GL(d,\mathbb{C})} \pi(g) f \, d\nu(g)$.

<u>Exercise</u>. 1) $\|\pi(g) f\|_2 = \|f\|_2$ for $f \in \mathcal{H}$,

2) $\|P_\nu f\|_2 \leq \|f\|_2$ for $f \in \mathcal{H}$.

3) $P_{\nu_1} P_{\nu_2} = P_{\nu_1 * \nu_2}$,

4) $P_\nu^* = P_{\nu^\vee}$, where $d\nu^\vee(g) = d\nu(g^{-1})$.

<u>Prop.</u> Under the assumptions of the theorem,

$$\|P_\nu\| < 1.$$

<u>Proof of Thm (assuming Prop.)</u>

Let $f(x) = \min\{C, \|x\|^{-\alpha}\}$ where $\alpha > 0$ is picked so that $f \in L^2(\mathbb{C}^d)$ and C will be specified later.

Let $K = \{x : 1 \leq \|x\| \leq 2\}$.

Let $\lambda = \|P_\nu\| < 1$.

We have $\overline{\lim\limits_{n\to\infty}} |\langle P_{\nu^{*n}} f, 1_K \rangle|^{1/n} = \overline{\lim\limits_{n\to\infty}} |\langle P_\nu^n f, 1_K \rangle|^{1/n}$

$$\leq \overline{\lim_{n\to\infty}} \|P_\nu^n\|^{1/n} \cdot \|f\|_2^{1/n} \cdot \|1_K\|_2^{1/n} \leq \lambda.$$

On the other hand,

$$\langle P_{\nu^{*n}} f, 1_K \rangle = \int_{1 \leq \|x\| \leq 2} \int_{GL(d,\mathbb{C})} \inf\{C, \|g^{-1} x\|^{-\alpha}\} \, d\nu^{*n}(g) \, dx$$

$$\geq \int_{1 \leq \|x\| \leq 2} \int_{GL(d,\mathbb{C})} \inf\{C; \|g^{-1}\|^{-\alpha} \cdot \|x\|^{-\alpha}\} \, d\nu^{*n}(g) \, dx$$

$$\geqslant \text{const} \cdot \int_{GL(d,\mathbb{C})} \inf\{c, \|g^{-1}\|^{-\alpha}\}\, d\nu^{*n}(g).$$

Since $\inf_{g \in SL_d(\mathbb{C})} \|g\| > 0$, we can choose c sufficiently large, so that $\inf\{c, \|g^{-1}\|^{-\alpha}\} = \|g^{-1}\|^{-\alpha}$.

We also use that $\|g^{-1}\| \leqslant \text{const} \cdot \|g\|^{d-1}$ (formula for inverse).

Hence, $\langle P_{\nu^{*n}} f, 1_K \rangle \geqslant \text{const} \cdot \int_{GL(d,\mathbb{C})} \|g\|^{-\alpha(d-1)} d\nu^{*n}(g).$

By Jensen's inequality,

$$\log \langle P_{\nu^{*n}} f, 1_K \rangle \geqslant \log(\text{const}) + \int_{GL(d,\mathbb{C})} \log\left(\|g\|^{-\alpha(d-1)}\right) d\nu^{*n}(g),$$

and

$$\int_{GL(d,\mathbb{C})} \log\|g\|\, d\nu^{*n}(g) \geqslant \frac{\log(\text{const})}{\alpha(d-1)} - \frac{1}{\alpha(d-1)} \log \langle P_{\nu^{*n}} f, 1_K \rangle$$

Therefore,

$$\lim_{n\to\infty} \frac{1}{n} \int_{GL(d,\mathbb{C})} \log\|g\| \cdot d\nu^{*n}(g) \geqslant -\frac{1}{\alpha(d-1)} \overline{\lim_{n\to\infty}} \frac{1}{n} \log \langle P_{\nu^{*n}} f, 1_K \rangle$$

$$\geqslant -\frac{1}{\alpha(d-1)} \cdot \log \lambda > 0.$$

<u>Proof of Prop.</u>

We have $P_\nu P_\nu^* = P_{\nu * \check{\nu}}$ a self-adjoint operator, and $\|P_\nu P_\nu^*\| = \|P_\nu\|^2$. Hence,

$$\|P_\nu\| < 1 \iff \|P_{\nu*\check{\nu}}\| < 1.$$

Without loss of generality, we may assume that P_ν is self-adjoint.

Suppose that $\|P_\nu\| = 1$.

Since $\|P_\nu\| = \sup_{\|f\|_2=1} |\langle P_\nu f, f \rangle|$, $\exists f_n : \|f_n\|_2 = 1$:

$$|\langle P_\nu f_n, f_n \rangle| \longrightarrow 1.$$

We have $|\langle P_\nu f_n, f_n \rangle| \leqslant \langle P_\nu |f_n|, |f_n| \rangle \leqslant 1$.

Hence, we may assume that $f_n \geqslant 0$.

$$\langle P_\nu f_n, f_n \rangle = \int_{G_\nu} \langle \pi(g) f_n, f_n \rangle \, d\nu(g) \longrightarrow 1.$$

Since $\langle \pi(g) f_n, f_n \rangle \leqslant 1$, it follows that

$$\langle \pi(g) f_n, f_n \rangle \to 1 \text{ for } \nu\text{-a.e. } g \in G_\nu.$$

We have $\| \pi(g) f_n - f_n \|_2^2 = 2 - 2 \cdot \langle \pi(g) f_n, f_n \rangle \to 0$,

and by Cauchy-Schwartz inequality,

$$\|\pi(g) f_n^2 - f_n^2\|_1 \leqslant \| \pi(g) f_n - f_n \|_2 \cdot \| \pi(g) f_n + f_n \|_2 \qquad (*)$$

$$\leqslant 2 \, \| \pi(g) f_n - f_n \|_2 \xrightarrow[n \to \infty]{} 0.$$

Consider the probability measures $\mu_n = f_n^2(x)\,dx$ on $\mathbb{C}^d$, and $\overline{\mu_n}$ on the projective space $\mathbb{P}(\mathbb{C}^d)$ which are projections of μ_n.

Since $\mathbb{P}(\mathbb{C}^d)$ is compact, $\{\overline{\mu_n}\}$ has a convergent subsequence: $\overline{\mu_{n_i}} \longrightarrow \overline{\mu}$.

It follows from (*) that $\bar{\mu}$ is g-invariant for ν-a.e. $g \in G_\nu$. Hence, $\bar{\mu}$ is G_ν-invariant. This gives a contradiction because of the following lemma. $\square$

<u>Lem.</u> Suppose that G_ν is totally irreducible and noncompact. Then there is no G_ν-inv. probability measure on $\mathbb{P}(\mathbb{C}^d)$.

<u>Proof.</u> Suppose that μ is a probabity G_ν-inv. measure.

Since G_ν is unbounded, $\exists g_n \in G_\nu : \|g_n\| \to \infty$.

Let $h_n = \frac{g_n}{\|g_n\|}$. Then $\det(h_n) = \|g_n\|^{-d} \to 0$.

Since $\|u_n\| = 1$, passing to a subsequence, we may assume that $u_n \to u \in M(d, \mathbb{C})$, $\|u\| = 1$, $\det(u) = 0$.

Let $V = [\mathrm{Ker}(u)] \subset \mathbb{P}(\mathbb{C}^d)$ and $W = [\mathrm{Im}(u)] \subset \mathbb{P}(\mathbb{C}^d)$.

We write $\mu = \mu_1 + \mu_2$ with $\mu_1 = \mu|_V$, $\mu_2 = \mu|_{V^c}$.

For $x \in V^c$, $g_n \cdot x = u_n \cdot x \to u \cdot x$.

Hence, $\mu = \lim_{n\to\infty} g_n \cdot \mu = \lim_{n\to\infty} g_n \cdot \mu_1 + u \cdot \mu_2$.

Passing to a subsequence, we may assume that

$g_n \mu_1 \to \mu_1^\infty$ – prob. measure on $\mathbb{P}(\mathbb{C}^d)$,

$g_n V \to V^\infty$ – projective subspace of $\mathbb{P}(\mathbb{C}^d)$.

Then $\mathrm{supp}(\mu_1^\infty) \subset V^\infty$, $\mathrm{supp}(u \cdot \mu_2) \subset W$, and

$$\mathrm{supp}(\mu) \subset V^\infty \cup W \Leftrightarrow \mu(V^\infty \cup W) = 1.$$

Let $F \subset V^\infty \cup W$ be the smallest union of projective subspaces such that $\mu(F) = 1$. Then $gF = F$, and this contradicts total irreducibility.

Spectral gap on the torus.

$\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$, $\Gamma \subset SL(d, \mathbb{Z})$

Fix a probability measure ν on Γ, $\Gamma = \langle \mathrm{supp}(\nu) \rangle$, and consider the averaging operator

$$P_\nu f(x) = \sum_{\gamma \in \Gamma} \mu(\gamma) \cdot f(\gamma^{-1} x), \quad f : \mathbb{T}^d \to \mathbb{C}.$$

<u>Thm.</u> Assume that Γ is totally irreducible and infinite. Then $\exists \lambda \in (0,1)$: $\| P_\nu f - \int_{\mathbb{T}^d} f \|_2 \leq \lambda \cdot \|f\|_2$ for $f \in L^2(\mathbb{T}^d)$.

<u>Proof.</u> Let $e_k(x) = e^{2\pi i \langle k, x \rangle}$, $k \in \mathbb{Z}^d$.

The Fourier transform $f \longmapsto \hat{f}$, $\hat{f}(k) = \langle f, e_k \rangle$ defines an isomorphism $L^2(\mathbb{T}^d)$ and $\ell^2(\mathbb{Z}^d)$

Under this isomorphism, the Γ-action on $L^2(\mathbb{T}^d)$ corresponds to the action $\gamma \cdot e_k = e_{{}^t\gamma^{-1} k}$.

Let $L^2_0(\mathbb{T}^d) = \{ f \in L^2(\mathbb{T}^d) : \int_{\mathbb{T}^d} f = 0 \}$.

Then $L^2_0(\mathbb{T}^d) \longleftrightarrow \ell^2(\mathbb{Z}^d \setminus \{0\})$.

We apply the argument of Prop. above to $\Gamma \curvearrowright \ell^2(\mathbb{Z}^d \setminus \{0\})$ (c.f. $\Gamma \curvearrowright L^2(\mathbb{R}^d)$).

This gives $\| P_\nu|_{\ell^2(\mathbb{Z}^d\setminus\{0\}) \to \ell^2(\mathbb{Z}^d\setminus\{0\})} \| < 1$, and Thm follows. $\square$

Rmk. Thm implies that

$$P_\nu^n f \xrightarrow{L^2} \int_{\mathbb{T}^d} f \quad \text{exponentially fast.}$$