Lecture 3: Multiplicative Ergodic Theorem.

Lyapunov Exponents.

$V =$ vector space.

$$V \wedge V = \langle v \wedge w : v, w \in V \rangle \Big/ \Big\langle \begin{array}{l} (\alpha v_1 + \beta v_2) \wedge w - \alpha (v_1 \wedge w) - \beta (v_2 \wedge w) \\ v \wedge w = - w \wedge v : \ v, w \in V, \ \alpha, \beta \in \mathbb{C} \end{array} \Big\rangle$$

If $\{e_i\}$ is a basis of V, then $\{e_i \wedge e_j\}_{i<j}$ is a basis of $V \wedge V$.

Given a linear map $A : V \to V$, we get a linear map $\Lambda^2 A : V \wedge V \longrightarrow V \wedge V$

$$v \wedge w \longmapsto Av \wedge Aw.$$

We fix orthonormal basis $\{e_i\}$ of V, and introduce a scalar product on $\Lambda^k V$ so that $\{e_{i_1} \wedge \ldots \wedge e_{i_k}\}_{i_1 < \ldots < i_k}$ is an orthonormal basis.

<u>Notation</u>: $\|v\|$ - Euclidean norm, $\|A\| = \sup_{\|v\|=1} \frac{\|Av\|}{\|v\|}$.

$\|A\| = \rho_1$ - largest eigenvalue of $(A^*A)^{1/2}$.

Let $\rho_1 \geq \ldots \geq \rho_d$ be eigenvalues of $(A^*A)^{1/2}$.

Then $\|\Lambda^k A\| = \rho_1 \cdots \rho_k$.

We also have: $\lim_{n \to \infty} \frac{\log \|\Lambda^k A^n\|}{n} = \log(\rho_1 \cdots \rho_k)$.

This observation allows to define Lyapunov exponents for products of random matrices.

Let $(\Omega, \mathbb{P})$ be a prob. space with $\theta: \Omega \to \Omega$ a measure-preserving map, and $X: \Omega \to GL(d, \mathbb{C})$ measurable with $\int_\Omega \log^+ \|X(\omega)\| \, d\omega < \infty$.

We set $S_n(\omega) = X(\theta^{n-1} \omega) \dots X(\omega)$.

<u>Questions.</u> 1) What are possible growth rates $\|S_n v\|$, $v \in \mathbb{C}^d$?

2) Does $S_n \simeq \Lambda^n$ for $\Lambda \in GL(d, \mathbb{C})$?

We define the <u>Lyapunov exponents</u> $\lambda_1(\omega), \dots \lambda_d(\omega)$ by

$$\boxed{\lambda_1(\omega) + \dots + \lambda_k(\omega) = \lim_{n \to \infty} \frac{\log \| \wedge^k S_n(\omega) \|}{n}.} \quad (*)$$

Note that this limit exists a.e. by Furstenberg - Kesten Thm.

<u>Multiplicative Ergodic Thm I. (Oseledets')</u>

Let $\rho_1^{(n)}(\omega) \geq \dots \geq \rho_d^{(n)}(\omega)$ be the eigenvalues of $(S_n^*(\omega) S_n(\omega))^{1/2}$. Then for a.e. ω,

$$\lim_{n \to \infty} \frac{\log \rho_i^{(n)}(\omega)}{n} = \lambda_i(\omega).$$

PROOF. Since $\|\Lambda^k S_n\| = \rho_1^{(n)} \cdots \rho_k^{(n)}$, this follows from $(*)$ $\square$

ME Thm II. For a.e. ω,

$$\lim_{n\to\infty} \left(S_n^*(\omega) S_n(\omega)\right)^{1/2n} \stackrel{def}{=} \Lambda(\omega)$$

exists, and the eigenvalues of Λ are e^{λ_i}.

Spaces of flags: Fix $0 = \tau_{s+1} < \tau_s < \ldots < \tau_1 = d$. A flag of type $\tau = (\tau_1, \ldots \tau_{s+1})$ is a sequence of subspaces $0 = V_{s+1} \subset V_s \subset \ldots \subset V_1 = \mathbb{C}^d$ such that $\dim V_i = \tau_i$. Let $\mathcal{F}(\tau)$ be the space of flags of type τ.

Fix $\sigma_1, \ldots \sigma_s, h > 0$, $\sigma_i \neq \sigma_j$ for $i \neq j$.

Given flags $V^{(1)} = \{V_{s+1}^{(1)} \subset \ldots \subset V_1^{(1)}\}$ and $V^{(2)} = \{V_{s+1}^{(2)} \subset \ldots \subset V_1^{(2)}\}$ we write $V_i^{(j)} = \mathcal{U}_i^{(j)} \perp V_{i+1}^{(j)}$

Then $\mathbb{C}^d = \mathcal{U}_1^{(j)} \perp \ldots \perp \mathcal{U}_s^{(j)}$.

Let
$$d(V^{(1)}, V^{(2)}) = \max_{\substack{i \neq j,\ \|x\| = \|y\| = 1 \\ x \in \mathcal{U}_i^{(1)},\ y \in \mathcal{U}_j^{(2)}}} |\langle x, y\rangle|^{h \cdot |\sigma_i - \sigma_j|^{-1}}.$$

Lem. Provided that $h^{-1}|\sigma_i - \sigma_j| \geq s-1$ for $i \neq j$, d defines a metric on $\mathcal{F}(\tau)$.

We prove Thm II under additional assumption:

$$\log^+ \|X^{-1}\| \in L^1.$$

Proof of Thm II. We fix ω for which Thm I holds, and set $X_n = X(\theta^{n-1}\omega)$.

By Lem. 4 (Lecture I), we may assume that

$$\overline{\lim_{n\to\infty}} \frac{\log \|X_n^{\pm 1}\|}{n} \leq 0. \qquad (*)$$

Let $\alpha_1 > \ldots > \alpha_s$ be different values of λ_i's.

Fix small $\varepsilon > 0$.

$\mathcal{U}_i^{(n)}$ = subspace generated by eigenvectors of $(S_n^* S_n)^{1/2}$ with eigenvalues $\rho_i^{(n)}$ satisfying

$$\left| \frac{\log \rho_i^{(n)}}{n} - \alpha_i \right| < \varepsilon.$$

$P_i^{(n)}$ = orthogonal projection on $\mathcal{U}_i^{(n)}$.

Claim. $\forall i \neq j$ $\forall$ sufficiently large n:

$$\| P_i^{(n)} P_j^{(n+1)} \| = \| P_j^{(n+1)} P_i^{(n)} \| \leq e^{(-|\alpha_i - \alpha_j| + 4\varepsilon)n}.$$

$x \in \mathbb{C}^d$, $y = P_i^{(n)} x \in \mathcal{U}_i^{(n)}$, $z = P_j^{(n+1)} y$.

Let $i > j$. Since $y \in \mathcal{U}_i^{(n)}$,

$$\|S_{n+1} y\| \leq \|X_{n+1}\| \cdot \|S_n y\| \leq \|X_{n+1}\| \cdot e^{(\alpha_i + \varepsilon) n} \|y\|.$$

Since $S_{n+1}^* S_{n+1}$ preserves the spaces $\mathcal{U}_k^{(n+1)}$, and $\mathcal{U}_{k_1}^{(n+1)} \perp \mathcal{U}_{k_2}^{(n+1)}$ for $k_1 \neq k_2$,

$$\langle S_{n+1} z, S_{n+1}(y-z) \rangle = \langle S_{n+1} z, S_{n+1} P_j^{(n+1)} (y-z) \rangle = 0.$$

Hence,

$$\|S_{n+1} y\| = \sqrt{\|S_{n+1} z\|^2 + \|S_{n+1}(y-z)\|^2} \geq \|S_{n+1} z\|$$
$$\geq e^{(\alpha_j - \varepsilon)(n+1)} \|z\| \geq e^{(\alpha_j - 2\varepsilon) n} \cdot \|z\|$$

and $\|z\| \leq \|X_{n+1}\| \cdot e^{(\alpha_i - \alpha_j + 3\varepsilon) n} \|y\|$.

By $(*)$, $\|X_{n+1}\| \leq e^{\varepsilon n}$ for $n \gg 0$.

Finally, $\|P_j^{(n+1)} P_i^{(n)} x\| \leq e^{(\alpha_i - \alpha_j + 4\varepsilon) n} \|P_i^{(n)} x\|$
$$\leq e^{(\alpha_i - \alpha_j + 4\varepsilon) n} \|x\|.$$

Let $i < j$. Then $x \in \mathbb{C}^d$, $y = P_j^{(n+1)} x$, $z = P_i^{(n)} y$.

$$\|S_n \cdot y\| = \|X_{n+1}^{-1} S_{n+1} y\| \leq \|X_{n+1}^{-1}\| \cdot \|S_{n+1} y\|$$
$$\leq \|X_{n+1}^{-1}\| \cdot e^{(\alpha_j + \varepsilon)(n+1)} \|y\| \leq e^{(\alpha_j + 3\varepsilon) n} \cdot \|y\|.$$

by $(*)$

$$\|S_n y\| = \left(\|S_n z\|^2 + \|S_n(y-z)\|^2\right)^{1/2} \geq \|S_n z\| \geq e^{(\alpha_i-\varepsilon)n}\|z\|.$$

Hence, $\|z\| \leq e^{(\alpha_j-\alpha_i+4\varepsilon)n} \cdot \|y\|$, and it follows that

$\|P_i^{(n)} P_j^{(n+1)}\| \leq e^{(\alpha_j-\alpha_i+4\varepsilon)n}$. Claim is proved.

Consider a sequence of flags

$$V^{(n)} = \{ V_{s_n+1}^{(n)} \subset \ldots \subset V_1^{(n)} \}$$

where $V_i^{(n)} = \bigoplus_{k=i}^{s_n} \mathcal{U}_k^{(n)}$.

For sufficiently large n, $V^{(n)} \in \mathcal{F}(\tau)$ for fixed τ.

By Claim, $\forall x \in \mathcal{U}_i^{(n)}$, $y \in \mathcal{U}_j^{(n+1)}$, $\|x\|=\|y\|=1$, $i \neq j$,

$$|\langle x, y\rangle| = |\langle x, P_i^{(n)} P_j^{(n+1)} y\rangle| \leq e^{(-|\alpha_i-\alpha_j|+4\varepsilon)n}.$$

We use a metric $\mathcal{F}(\tau)$ with $\sigma_i = \alpha_i$.

Then $d(V^{(n)}, V^{(n+1)}) \leq \max_{i\neq j} e^{(-|\alpha_i-\alpha_j|+4\varepsilon)n \cdot h\, |\alpha_i-\alpha_j|^{-1}}$

$$\leq e^{-(1-\varepsilon')hn}, \quad \varepsilon' \approx 0.$$

This implies that the sequence is fundamental.

Hence, $V^{(n)} \to V^\infty \in \mathcal{F}(\tau)$, and

$$d(V^{(n)}, V^\infty) \leq \text{const} \cdot e^{-(1-\varepsilon')hn}. \qquad (**)$$

We have $(S_n^* S_n)^{1/2n} = \sum_j (\rho_j^{(n)})^{1/n} \cdot Q_j^{(n)}$, where

$(\rho_j^{(n)})^{1/n} \to \alpha_i$ (by Thm I)

projection on $\rho_j^{(n)}$ - eigenspace

$\sum_j Q_j^{(n)} \to P_i$ (by (**)). Hence, $(S_n^* S_n)^{1/2n} \to \sum_i \alpha_i P_i$.

ME Thm III: For a.e. ω, there exists a flag $\mathcal{V}(\omega) = \{V_{s+1} \subset \ldots \subset V_1\}$ such that $\forall x \in V_i \setminus V_{i+1}$:

$$\lim_{n\to\infty} \frac{\log \|S_n(\omega)x\|}{n} = \alpha_i(\omega)$$

where $\alpha_1(\omega) > \ldots > \alpha_s(\omega)$ are different values of Lyapunov exponents $\lambda_i(\omega)$'s.

Proof. We take $V(\omega)$ to be the flag $\mathcal{V}^\infty$ construted above. We write $V_i = \mathcal{U}_i \perp V_{i+1}$.

Let $x \in \mathcal{U}_i$, $\|x\| = 1$, $x_j^{(n)} = P_j^{(n)} x$.

By (**), for $j \neq i$, $\|x_j^{(n)}\|^{h|\alpha_i - \alpha_j|^{-1}} \leq \text{const} \cdot e^{-(1-\varepsilon')hn}$,

and $\|x_j^{(n)}\| \leq \text{const} \cdot e^{-(1-\varepsilon')|\alpha_i - \alpha_j| \cdot n}$.

Recall that $\mathcal{U}_i^{(n)} \perp \mathcal{U}_j^{(n)}$, $i \neq j$, $S_n^* S_n (\mathcal{U}_i^{(n)}) \subset \mathcal{U}_i^{(n)}$.

Hence, $\|S_n x\| = \left(\sum_j \|S_n x_j^{(n)}\|^2\right)^{1/2}$

$\leq \left(\sum_j e^{2(\alpha_j + \varepsilon)n} \|x_j^{(n)}\|^2\right)^{1/2}$, for $n \geq n_0(\varepsilon)$

$$\leq \Big(\sum_{j \geq i} e^{2(\alpha_j+\varepsilon)n} + \sum_{j<i} e^{2(\alpha_j+\varepsilon)n} \cdot e^{2(-1+\varepsilon')(\alpha_i-\alpha_j)n} \Big)^{1/2}$$

$$\leq \text{const} \cdot e^{(\alpha_i+\varepsilon'')n}, \qquad \varepsilon'' \approx 0.$$

This shows that

$$\overline{\lim_{n\to\infty}} \frac{\log \|S_n x\|}{n} \leq \alpha_i.$$

On the other hand,

$$\|S_n x\| \geq \|S_n x_i^{(n)}\| \geq e^{(\alpha_i-\varepsilon)n} \|x_i^{(n)}\| \quad \text{for } n \geq n_0(\varepsilon).$$

Since $x_i^{(n)} = P_i^{(n)} x \to P_i x \neq 0$,

$$\underline{\lim_{n\to\infty}} \frac{\log \|S_n x\|}{n} \geq \alpha_i.$$

<u>Proof of Lemma.</u>

We need to show that

$$d(V^{(1)}, V^{(2)}) \leq d(V^{(1)}, V^{(3)}) + d(V^{(3)}, V^{(1)}), \qquad V^{(1)}, V^{(2)}, V^{(3)} \in \mathcal{F}(\tau).$$

Take $x \in U_i^{(1)}$, $y \in U_j^{(2)}$, $\|x\| = \|y\| = 1$ such that

$$d(V^{(1)}, V^{(2)}) = |\langle x, y\rangle|^{h|\sigma_i - \sigma_j|^{-1}}.$$

Let $x = \sum_k x_k$, $y = \sum_k y_k$ where x_k and y_k are corresponding projections on $U_k^{(3)}$.

Then $|\langle x, y\rangle| \leqslant \sum_K \|x_K\| \cdot \|y_K\|$.

Let $d_i = d(V^{(i)}, V^{(3)})$. Then

$$\|x_K\| \leqslant d_1^{h^{-1}|\sigma_i - \sigma_K|}, \qquad \|y_K\| \leqslant d_2^{h^{-1}\cdot|\sigma_j - \sigma_K|}.$$

Hence, $d(V^{(1)}, V^{(2)}) \leqslant \left(\sum_{k=1}^{s} d_1^{h^{-1}|\sigma_i-\sigma_K|} \cdot d_2^{h^{-1}|\sigma_j-\sigma_K|}\right)^{h|\sigma_i-\sigma_j|^{-1}}$,

and it remains to show that

$$\sum_{k=1}^{s} d_1^{h^{-1}|\sigma_i-\sigma_K|} \cdot d_2^{h^{-1}|\sigma_j-\sigma_K|} \leqslant (d_1+d_2)^{h^{-1}|\sigma_i-\sigma_j|}.$$

Suppose that $d_2 \leqslant d_1$ and $d_2 = z d_1$ with $z \leq 1$.

Then we get:

$$\sum_{k=1}^{s} d_1^{h^{-1}(|\sigma_i-\sigma_K|+|\sigma_j-\sigma_K|)} \cdot z^{h^{-1}|\sigma_K-\sigma_j|} \leqslant (1+z)^{h^{-1}|\sigma_i-\sigma_j|} \cdot d_1^{h^{-1}|\sigma_i-\sigma_j|}.$$

Since $d_1 \leq 1$, $d_1^{h^{-1}(|\sigma_i-\sigma_K|+|\sigma_j-\sigma_K|)} \leqslant d_1^{h^{-1}|\sigma_i-\sigma_j|}$.

Now we have to check that:

$$\sum_{k=1}^{s} z^{h^{-1}|\sigma_k-\sigma_j|} \leqslant (1+z)^{h^{-1}|\sigma_i-\sigma_j|}.$$

Using the assumption,

$$\sum_{k=1}^{s} z^{h^{-1}|\sigma_k-\sigma_j|} \leqslant 1+(s-1)z^{s-1} \leqslant (1+z)^{s-1} \leqslant (1+z)^{h^{-1}|\sigma_i-\sigma_j|}.$$

$\square$