Lecture 4: Stationary measures.

X = compact metric space (e.g. X = proj. space).

$G < GL(d, \mathbb{C})$ - closed subgroup acting on X.

μ = probability measure on G.

Def. A prob. measure ν on X is *μ-stationary* if $\mu * \nu = \nu$, that is,

$$\forall f \in C(X): \int_{G \times X} f(g \cdot x)\, d\mu(g)\, d\nu(x) = \int_G f\, d\nu.$$

Prop. $\exists$ μ-stationary measure.

Proof. Recall that the space $\mathcal{P}(X)$ of prob. measures is compact. Consider the sequence

$$\nu_n = \frac{1}{n} \sum_{i=0}^{n-1} \mu^{*i} * \nu.$$

We have

$$\mu * \nu_n - \nu_n = \frac{1}{n}\left(\sum_{i=0}^{n-1} \mu^{*i+1} * \nu_n - \sum_{i=0}^{n-1} \mu^{*i} * \nu_n \right) = \frac{\mu^{*n} * \nu_n - \nu_n}{n} \to 0.$$

By compactness, $\nu_{n_k} \to \nu \in \mathcal{P}(X)$, and $\mu * \nu = \nu$. $\square$

Rmk. If G is strongly irreducible and noncompact, then $\nexists$ G-inv. measures on $P = \mathbb{P}(\mathbb{C}^d)$.

Question. When is the stationary measure unique?

Def. A set $S \subset GL(d,\mathbb{C})$ is <u>contracting</u> if $\exists$ sequence $S_n \in S$: $\frac{S_n}{\|S_n\|} \longrightarrow$ rank-1 matrix.

Notation: S_μ = closed semigroup generated by $supp(\mu)$.

Thm. 1 Assume that S_μ is
- totally irreducible
- contracting.

Then the μ-stationary measure is unique.

Thm. 2 With notation as in Thm. 1, if $\int \log^+\|g\| \, d\mu(g) < \infty$, the top Lyapunov exponent λ is given by

$$\lambda = \int_{G\times P} \log \frac{\|gx\|}{\|x\|} \, d\mu(g) \, d\nu(x)$$

where ν is the μ-stationary measure.

Prop. 1. Let μ be a prob. measure on $GL(d,\mathbb{C})$ such that S_μ is totally irreducible, and ν a μ-stationary measure. Then

$$\nu(V) = 0 \text{ for any proper subspace } V \subset \mathbb{P}.$$

Def. We call such ν <u>proper</u>.

Proof. Let ℓ be the minimal dimension of subspace $V \subset \mathbb{P}$ such that $\nu(V) > 0$.

Let $\Sigma_{\geq \varepsilon} = \left\{ V\text{-subspace of } \mathbb{P} : \begin{array}{l} \dim(V) = \ell \\ \nu(V) \geq \varepsilon \end{array} \right\}$.

Let $V_1, \dots V_k$ be distinct elements of $\Sigma_{\geq \varepsilon}$.

Then $\dim(V_i \cap V_j) < \ell$, $i \neq j$, and $\nu(V_i \cap V_j) = 0$.

Hence, $\nu(V_1 \cup \dots V_k) = \sum_{i=1}^{k} \nu(V_i) \geq k \cdot \varepsilon$.

This shows that $\Sigma_{\geq \varepsilon}$ is finite.

Let $V \in \Sigma_{\geq \varepsilon}$ with $\nu(V)$ maximal.

$$\nu(V) = \int_{GL(d,\mathbb{C}) \times \mathbb{P}} 1_V(gx)\, d\mu(g)\, d\nu(x) = \int_{GL(d,\mathbb{C})} \nu(g^{-1}V)\, d\mu(g).$$

Since $\nu(g^{-1}V) \leq \nu(V)$, it follows that $\nu(g^{-1}V) = \nu(V)$ for μ-a.e. g.

Let L be the union of V with $\nu(V)$ maximal.

It follows that $S_\mu \cdot L = L$.

Hence, $\Sigma_{>0} = \emptyset$. $\blacksquare$

Recall that $\Omega = GL(d,\mathbb{C})^{\mathbb{N}}$ equipped with the product measure $\mathbb{P} = \mu^{\otimes \mathbb{N}}$.

Given a stationary measure ν and $f \in C(\mathbb{P})$, we consider $\varphi_n : \Omega \to \mathbb{C}$ defined by

$$\varphi_n(w) = \int_P f(w_1 \dots w_n x)\, d\nu(x).$$

Notation (conditional expectation)

$\mathcal{F}$ = sub-σ-algebra

$E(\cdot|\mathcal{F}) : L^2(\Omega) \longrightarrow L^2(\Omega)$

orthogonal projection on the subspace of of $\mathcal{F}$-measurable functions.

We consider a sequence of σ-algebras:

$$\mathcal{F}_n = \{ A \times (\prod_{i \geq n+1} GL(d, \mathbb{C})) : \underbrace{A \subset \prod_{i=1}^{n} GL(d, \mathbb{C})}_{\text{measurable}} \}$$

Exercise: $E(\varphi, \mathcal{F}_n) = \int_\Omega \varphi(w) \prod_{i \geq n+1} d\mu(w_i)$.

We observe that $\{\varphi_n\}$ satisfy the following "martingale property":

$$E(\varphi_{n+1} | \mathcal{F}_n) = \int_{GL(d,\mathbb{C}) \times P} f(w_1 \dots w_{n+1} x)\, d\mu(w_{n+1})\, d\nu(x)$$

$$= \int_P f(w_1 \dots w_n x)\, d\nu(x) = \varphi_n,$$

where we used that ν is stationary.

Martingale convergence theorem.

$(\Omega, \mathbb{P})$ – probability space

$\varphi_n : \Omega \to \mathbb{C}$ – measurable functions

$\mathcal{F}_n$ – increasing sequence of σ-algebras.

Def. $\{\varphi_n\}$ is a martingale (w.r.t. $\{\mathcal{F}_n\}$) if

(1) φ_n is $\mathcal{F}_n$-measurable,

(2) $E(\varphi_{n+1} | \mathcal{F}_n) = \varphi_n$.

Exercise. Show that $E(\varphi_{n+k} | \mathcal{F}_n) = \varphi_n$ for $k \geq 0$,

and $E(\varphi_{n+1}) = E(\varphi_n)$.

Lem. Given a function $\tau : \Omega \to \mathbb{N}$ such that $\{\tau \leq n\} \in \mathcal{F}_n$, the sequence $\{\varphi_{\min\{n, \tau(\omega)\}}(\omega)\}$ is a martingale.

Proof. $E(\varphi_{\min\{n+1, \tau\}} | \mathcal{F}_n) = E(\varphi_\tau \cdot 1_{\{\tau \leq n\}} + \varphi_{n+1} \cdot 1_{\{\tau > n\}} | \mathcal{F}_n)$

$$= \varphi_\tau \cdot 1_{\{\tau \leq n\}} + 1_{\{\tau > n\}} E(\varphi_{n+1} | \mathcal{F}_n)$$

$$= \varphi_\tau \cdot 1_{\{\tau \leq n\}} + 1_{\{\tau > n\}} \varphi_n = \varphi_{\min\{n, \tau\}}.$$

∎

Def. $\{\varphi_n\}$ – a sequence of numbers, (a,b) – interval

The upcrossing number

$u_n(a,b)$ = number of times $0 < i \leq n$ that sequence φ_n goes from $(-\infty, a)$ to $(b, +\infty)$.

<u>Exercise.</u> $\{\varphi_n\}$ converges (possibly to $\pm\infty$)

$$\Updownarrow$$

$$\forall (a,b):\ u_\infty(a,b) < \infty$$

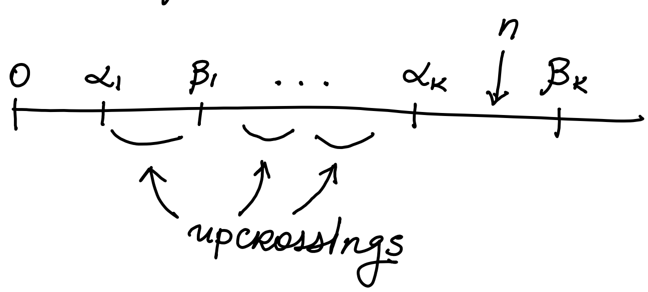
$$\Updownarrow$$

$$\forall (a,b),\ a,b \in \mathbb{Q},\quad u_\infty(a,b) < \infty.$$

<u>Lem.</u> $E(u_n(a,b)) \leq \dfrac{E|\varphi_n| + |a|}{b-a}$.

<u>Proof.</u> $\alpha_0 = \beta_0 = 0$, $\alpha_n = \min\{i > \beta_{i-1} : \varphi_i \leq a\}$,

$\beta_n = \min\{ i > \alpha_n : \varphi_i \geq b\}$.



Consider $\psi_n = \sum_{i \geq 1} (\varphi_{\min\{n,\beta_i\}} - \varphi_{\min\{n,\alpha_i\}})$.

We have

$$\psi_n = \underbrace{(\varphi_{\beta_1} - \varphi_{\alpha_1})}_{\geq b-a} + \dots + \underbrace{(\varphi_{\beta_{k-1}} - \varphi_{\alpha_{k-1}})}_{\geq b-a} + (\varphi_n - \varphi_{\alpha_k}) + 0$$

$$\geq u_n(a,b)\cdot(b-a) + (\varphi_n - a)$$

Hence, $E[\psi_n] \geq (b-a)\cdot E[u_n(a,b)] - E|f_n| - |a|$.

Since $\{\alpha_n \leq k\}$ depends only on φ_i, $i \leq k$,

by Lem., ψ_n is a martingale.

Hence, $E[\psi_n] = E[\psi_0] = 0$. $\blacksquare$

Martingale convergence Thm.
Let $\{\varphi_n\}$ be a martingale with $\sup_n E|\varphi_n| < \infty$.
Then $\varphi_n(w)$ converges for a.e. w.

Proof. By Lemma and monotone convergence Thm,
$E[u_\infty(a,b)] < \infty$ for every interval (a,b).
In particular, $\mathbb{P}(u_\infty(a,b) < \infty) = 1$, and
$\mathbb{P}(u_\infty(a,b) < \infty \ \forall \text{rational } (a,b)) = 1$.
This implies convergence a.e.

Convergence in the space of measures.

μ = prob. measure on $GL(d,\mathbb{C})$
$\Omega = GL(d,\mathbb{C})^{\mathbb{N}}$ with $\mathbb{P} = \mu^{\otimes \mathbb{N}}$.

Prop. 2 Let ν be a μ-stationary measure on $\mathbb{P}$.
For a.e. $w = (w_n) \in \Omega$, $\exists$ a prob. measure ν_w on $\mathbb{P}$
such that $w_1 \dots w_n \cdot g\nu \xrightarrow[n\to\infty]{} \nu_w$
for a.e. $g \in GL(d,\mathbb{C})$ with respect to
$\lambda = \sum_{n=0}^{\infty} 2^{-n-1}\mu^{*n}$. The measures ν_w satisfy

$$\int_{\mathbb{P}} f\,d\nu = \int_\Omega \left(\int f\,d\nu_w \right) dw.$$

Proof. Let $f \in C(P)$ and $F(g) = \int_P f(gx)\, d\nu(x)$.

As we observed above, $\varphi_n(w) = F(w_1 \cdot \ldots \cdot w_n)$ is a martingale. By the Martingale Conv. Thm, $\varphi_n(w) \to \varphi_\infty(w)$ for a.e. $w \in \Omega$.

Since f and φ_n are uniformly bounded,

$$\int_\Omega \varphi_\infty(w)\, dw = \lim_{n\to\infty} \int_\Omega \varphi_n(w)\, dw = \int_\Omega \varphi_1(w)\, dw =$$

$$= \int_{GL(d,\mathbb{C})\times P} f(w_1 \cdot x)\, d\mu(w_1) d\nu(x) = \int_P f\, d\nu.$$

For $k, r \geq 0$,

$$\|\varphi_{k+r} - \varphi_k\|_2^2 = \|\varphi_{k+r}\|_2^2 + \|\varphi_k\|_2^2 - 2\langle \varphi_{k+r}, \varphi_k \rangle, \text{ and}$$

$$\langle \varphi_{k+r}, \varphi_k \rangle = \int_{\Omega\times P^2} f(w_1 \ldots w_{k+r} x)\, \overline{f}(w_1 \ldots w_k y)\, dw\, d\nu(x) d\nu(y)$$

$$\underset{\text{by stationarity}}{=} \int_{\Omega\times P^2} f(w_1 \ldots w_k x)\, \overline{f}(w_1 \ldots w_k y)\, dw\, d\nu(x) d\nu(y)$$

$$= \|\varphi_k\|_2^2.$$

(this also follows from the martingale property.)

Hence, $\|\varphi_{k+r} - \varphi_k\|_2^2 = \|\varphi_{k+r}\|_2^2 - \|\varphi_k\|_2^2$, and

$$\sum_{k=1}^{n} \|\varphi_{k+r} - \varphi_k\|_2^2 < \mathrm{const}(f) \cdot r \quad (*)$$

For $\omega \in \Omega$, we set, $\omega_{\leq n} = \omega_1 \dots \omega_n$

Then

$$\sum_{K=1}^{\infty} \int_{\Omega \times GL(d,\mathbb{C})} |F(\omega_{\leq K} \cdot g) - F(\omega_{\leq K})|^2 \, d\omega \, d\lambda(g)$$

$$= \sum_{K=1}^{\infty} \sum_{r=0}^{\infty} 2^{-r-1} \int_{\Omega \times GL(d,\mathbb{C})} |F(\omega_{\leq K} \cdot g) - F(\omega_{\leq K})|^2 \, d\omega \, d\mu^{*r}(g)$$

$$= \sum_{K=1}^{\infty} \sum_{r=0}^{\infty} 2^{-r-1} \int_{\Omega} |F(\omega_{\leq K+r}) - F(\omega_{\leq K})|^2 \, d\omega \underset{\text{by } (*)}{<} \infty$$

This shows that

$$\sum_{K=1}^{\infty} |F(\omega_{\leq K} \cdot g) - F(\omega_{\leq K})|^2 < \infty$$

for $(\mathbb{P} \times \lambda)$-a.e. (ω, g), and

$$\lim_{K \to \infty} F(\omega_{\leq K} \cdot g) = \lim_{K \to \infty} F(\omega_{\leq K}) = \varphi_\infty(\omega).$$

This proves convergence for fixed $f \in C(P)$.
Since $C(P)$ is seperable, for a set of full measure of (ω, g),

$$\int_P f(\omega_{\leq K} \cdot g x) \, d\nu(x) \longrightarrow \varphi_\infty^f(\omega) \quad \forall f \in C(P).$$

Every limit point $\nu_{\omega,g}$ of $\omega_{\leq K} \cdot g \cdot \nu$ satisfies

$$\int_P f \, d\nu_{\omega,g} = \varphi_\infty^f(\omega).$$

Hence, $\nu_{\omega,g}$ is independent of g, and
$$\lim_{k\to\infty} \omega_{\le k}\, g\cdot\nu = \nu_\omega .$$
Since $\int_\Omega \varphi_\infty^f(\omega)\,d\omega = \int_P f\,d\nu$, the last equality follows. $\lrcorner$

<u>Proof of Thm. 1.</u> Let ν be a μ-stationary measure. By Prop. 2, for a.e. ω and λ-a.e. g,
$$\omega_{\le n}\cdot g\cdot\nu \xrightarrow[n\to\infty]{} \nu_\omega .$$
We fix such ω. Passing to a subsequence,
$$\frac{\omega_{\le n_i}}{\|\omega_{\le n_i}\|} \xrightarrow[i\to\infty]{} A\in M(d,\mathbb{C}).$$
For $v\in P$, $v\notin \mathrm{Ker}(A_\omega)$, $\omega_{\le n_i}\cdot v \longrightarrow A_\omega v$.

By Prop. 1, ν (and $g\cdot\nu$) is proper.

Hence, $\omega_{\le n_i}\cdot g\cdot\nu \xrightarrow[i\to\infty]{} A\cdot g\cdot\nu$, and
$$A\cdot g\cdot\nu = \nu_\omega .$$
This holds for $g\in \mathrm{supp}(\lambda) = S_\mu$.

Let $\{g_n\}\subset S_\mu$ be a contracting sequence,
$$\frac{g_n}{\|g_n\|}\longrightarrow B, \quad \mathrm{rank}(B)=1.$$

Since S_μ is irreducible, we may arrange that $\mathrm{Im}(B) \not\subset \mathrm{Ker}(A_\omega)$

Again since ν is proper,

$$g_n \nu \longrightarrow B \cdot \nu.$$

Since $\mathrm{rank}(B)=1$, $B\cdot\nu$ is a Dirac measure.

Also, $\nu_\omega = A \cdot B \nu$ is Dirac.

In fact, $\nu_\omega = \delta_{x(\omega)}$ where $x(\omega) = [A(\mathrm{Im}(B))]$.

Since $\int_P f \, d\nu = \int_\Omega \left(\int_P f \, d\nu_\omega \right) d\omega$,

this shows that ν is unique.

(Note that A is independent of ν).

$\blacksquare$

$\boxed{\text{Cor. 1. Assume that } S_\mu \text{ is strongly irreducible and contracting. Then for every proper measure } \eta \text{ on } P \text{ and a.e. } \omega \in \Omega, \quad \omega_{\leq n} \cdot \eta \xrightarrow[n\to\infty]{} \delta_{x(\omega)}.}$

Rmk. "Typical" random products tend to become rank-1.

Proof. Passing to a subsequence,

$$\frac{w_{\leq n_i}}{\|w_{\leq n_i}\|} \to A \in M(d, \mathbb{C}).$$

In the above proof, we showed that $A \cdot \nu = \delta_{x(\omega)}$, i.e., $\nu(A^{-1}(x(\omega))) = 1$. Since ν is proper, this shows that A is rank-1.

Since η is proper, $w_{\leq n_i} \cdot \eta \to A \cdot \eta = \delta_{x(\omega)}$.

Here $\delta_{x(\omega)} = \nu_\omega$ does not depend on the chosen subsequence, and this implies convergence. $\square$

Cor. 2. Assume that S_μ is totally irreducible and contracting. Then

1) $\forall$ a.e. $\omega \in \Omega$: all limit points C of $\frac{S_n(\omega)}{\|S_n(\omega)\|}$ have rank-1, $\mathrm{Im}(C^*) = \langle x(\omega) \rangle$.

2) $\forall v \neq 0$: $\forall$a.e. $\omega \in \Omega$: $\|S_n(\omega)\| \leq c(\omega, v) \cdot \|S_n(\omega) v\|$.

Proof. Since $S_n(\omega)^* = \omega_1^* \dots \omega_n^*$, (1) follows from the proof of above corollary.

To prove (2), we write $S_n = k_1^{(n)} a^{(n)} k_2^{(n)}$ with $k_1^{(n)}, k_2^{(n)} \in SU(n)$ and $a^{(n)} = \mathrm{diag}(a_1^{(n)}, \dots a_d^{(n)})$

$a_1^{(n)} \geq \dots \geq a_d^{(n)} > 0$ (the Cartan decomposition).

Then $\| S_n \| = $ (max. eigenvalue of $S_n^* S_n$) $= a_1^{(n)}$, and

$$\| S_n v \|^2 = \sum_i (a_i^{(n)})^2 |\langle k_2^{(n)} v, e_i \rangle|^2 = (a_1^{(n)})^2 |\langle k_2^{(n)} v, e_1 \rangle|^2 - O((a_2^{(n)})^2).$$

It follows from (1) that $\frac{a_2^{(n)}}{a_1^{(n)}} \to \infty$.

Hence, it remains to show that for a.e. ω,

$$\underline{\lim}_{n\to\infty} |\langle k_2^{(n)}(\omega) v, e_1 \rangle| > 0. \quad (*)$$

Passing to a subsequence, $k_i^{(n)} \longrightarrow k_i \in SU(n)$,

and $\frac{S_n^*}{\| S_n \|} \to k_2^{-1} \begin{pmatrix} 1 & & 0 \\ & 0 & \\ 0 & & 0 \end{pmatrix} k_1^{-1} \overset{\text{def}}{=} A_\omega$.

Note that $Im(A_\omega) = \langle k_2^{-1} e_1 \rangle$.

Along this subsequence, $\langle k_2^{(n)} v, e_1 \rangle \to \langle v, k_2^{-1} e_1 \rangle$.

Let $X_\omega = [Im(A_\omega)] \subset P$.

Hence, $(*)$ would follow if we show that for a.e. ω, $\langle v, X_\omega \rangle \neq 0$.

By Prop. 2 (recall that $\nu_\omega = \delta_{x(\omega)}$),

$$\mathbb{P}(\{\omega : \langle v, X_\omega \rangle = 0\}) = \nu(\langle v \rangle^\perp) = 0$$

because ν is proper. $\square$

Rmk. The same proof also shows that in Cor. 2 (2), the estimate is uniform over v in compact subsets of $\mathbb{C}^d \setminus \{0\}$.

Proof of Thm. 2.

Consider the map $\sigma : GL(d,\mathbb{C}) \times P \to P$

$$(g, x) \longmapsto \log \frac{\|gx\|}{\|x\|}.$$

It satisfies the cocycle property:

$$\sigma(g_1 g_2, x) = \sigma(g_1, g_2 x) + \sigma(g_2, x).$$

We build a skew-product dynamical system:

on $(\widetilde{P}, \widetilde{\nu})$, where $\widetilde{P} = \Omega \times P$, $\widetilde{\nu} = \mathbb{P} \otimes \nu$.

$T : \widetilde{P} \to \widetilde{P} : ((\omega_i), x) \longmapsto ((\omega_{i+1}), \omega_1 \cdot x)$.

Since ν is stationary, T is measure-preserving.

Let $f(\omega, x) = \sigma(\omega_1, x)$.

By the cocycle property,

$$\begin{aligned}
\sigma(\omega_n \dots \omega_1, x) &= \sigma(\omega_n, (\omega_{n-1} \dots \omega_1) x) + \sigma(\omega_{n-1} \dots \omega_1, x) \\
&= \sum_{k=1}^{n} \sigma(\omega_k, (\omega_{k-1} \dots \omega_1) x) \\
&= \sum_{k=0}^{n-1} f(T^k \cdot (\omega, x)).
\end{aligned}$$

By the Pointwise Ergodic Thm, for a.e. (ω, x),

$$\frac{1}{n} \sigma(\omega_n \dots \omega_1, x) \xrightarrow[n \to \infty]{} \sigma_\infty(\omega, x),$$

$$\int \sigma_\infty \, d\widetilde{\nu} = \int f \, d\widetilde{\nu} = \int_{GL(d,\mathbb{C}) \times P} \log \frac{\|gx\|}{\|x\|} \cdot d\mu(g) \, d\nu(x).$$

On the other hand,

$$\int \sigma(w_n \dots w_1, x)\, dw = \int_{GL(d,\mathbb{C})^n} \log \frac{\|w_n \dots w_1 x\|}{\|x\|}\, d\mu(w_1) \dots d\mu(w_n)$$

$$= \int_{GL(d,\mathbb{C})} \log \frac{\|gx\|}{\|x\|}\, d\mu^{*n}(g).$$

By Cor. 2 (2),

$$\lim_{n\to\infty} \frac{1}{n} \int \sigma(w_n \dots w_1, x)\, dw = \lim_{n\to\infty} \frac{1}{n} \int_{GL(d,\mathbb{C})} \log \|g\|\, d\mu^{*n}(g).$$

The last limit is the top Lyapunov exponent λ (Lecture 2).

Hence, $$\lambda = \int_{GL(d,\mathbb{C}) \times P} \log \frac{\|gx\|}{\|x\|}\, d\mu(g)\, d\nu(x).$$