Lecture 5: Simplicity of Lyapunov spectrum.

μ = prob. measure on $GL(d,\mathbb{C})$, $\int_{GL(d,\mathbb{C})} \log^+ \|g\|\, d\mu(g) < \infty$.

$\Omega = GL(d,\mathbb{C})^{\mathbb{N}}$ with $\mathbb{P} = \mu^{\otimes \mathbb{N}}$,

$X : \Omega \to GL(d,\mathbb{C}) : \omega \mapsto \omega_1$, $\theta : \Omega \to \Omega : (\omega_n) \mapsto (\omega_{n+1})$

$$S_n(\omega) = X(\theta^{n-1}\omega) \dots X(\omega)$$

Recall that the Lyapunov exponents $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_d$ are defined by (see Lecture III):

$$\lambda_1 + \dots + \lambda_k = \lim_{n\to\infty} \frac{\log \|\Lambda^k S_n(\omega)\|}{n}.$$

(In this case, λ_i's are constant a.e.)

Thm. 1 (Guivarc'h – Raugi)
Suppose that S_μ is strongly irreducible. Then:
$$\lambda_1 > \lambda_2 \iff S_\mu \text{ is contracting.}$$

Proof. $\Longrightarrow$ Write $S_n = k_1^{(n)} a^{(n)} k_2^{(n)}$ where $k_1^{(n)}, k_2^{(n)} \in SU(n)$ and $a^{(n)} = \operatorname{diag}(a_1^{(n)}, \dots a_d^{(n)})$, $a_1^{(n)} \geq \dots \geq a_d^{(n)} > 0$.

Then $\|S_n\| = (\text{max. eigenvalue of } (S_n^* S_n)^{1/2}) = a_1^{(n)}$,

$$\|\Lambda^2 S_n\| = (\text{max eigenvalue of } \Lambda^2 (S_n^* S_n)^{1/2}) = a_1^{(n)} a_2^{(n)}.$$

Hence, $\lambda_i = \lim_{n\to\infty} \frac{\log a_i^{(n)}}{n}$. If $\lambda_1 > \lambda_2$, then

$$\lim_{n\to\infty} \frac{\log (a_2^{(n)}/a_1^{(n)})}{n} < 0 \quad \text{and} \quad \frac{a_2^{(n)}}{a_1^{(n)}} \to 0.$$

(for a.e. ω)

This implies that limit points of $\frac{S_n}{\|S_n\|}$ are rank-1.

Lem. Let $T:\Omega\to\Omega$ be a measure-preserving map of a prob. space $(\Omega,\mathbb{P})$. Let $f:\Omega\to\mathbb{R}$ be such that

$$\int_\Omega f^+<\infty \text{ and } \sum_{k\geq 1} f\circ T^k \overset{a.e.}{=} +\infty.$$

Then $$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1} f\circ T^k \overset{a.e.}{>} 0. \quad (*)$$

Proof. By Pointwise Ergodic Thm,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1} f\circ T^k \overset{a.e.}{=} f_\infty,$$

where f_∞ is a T-inv. function.

According to our assumption, $f_\infty \overset{a.e.}{\geq} 0$.

Let $\Omega_\infty=\{f_\infty=0\}$. Then on Ω_∞,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1} f\circ T^k \overset{a.e.}{=} 0.$$

Let $I_\varepsilon(t)=[t-\varepsilon,t-\varepsilon]$, $f_n(\omega)=\sum_{k=0}^{n-1} f(T^k\omega)$, and

$$R_n^\varepsilon(\omega)=\left|\bigcup_{i=1}^{n} I_\varepsilon(f_n(\omega))\right|.$$

Then $\forall\omega\in\Omega_\infty\ \forall\delta>0$: $\forall n\geq n_0(\omega,\delta)$: $|f_n(\omega)|\leq n\delta$.

Hence, $I_\varepsilon(f_n(\omega))\subset I_{2n\delta+\varepsilon}(0)$, and $R_n^\varepsilon(\omega)\leq R_{n_0}^\varepsilon(\omega)+2n\delta+\varepsilon$.

Hence, $\overline{\lim_{n\to\infty}} \frac{1}{n} R_n^\varepsilon(\omega) \leq 2\delta$ for all $\delta > 0$, and

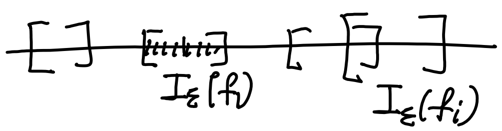
$$\frac{1}{n} R_n^\varepsilon \xrightarrow{a.e.} 0, \qquad \frac{1}{n}\int_{\Omega_\infty} R_n^\varepsilon \longrightarrow 0.$$

Since $f_n \circ T = f_{n+1} - f_1$, we have

$$R_{n+1}^\varepsilon - R_n^\varepsilon \circ T = \Big|\bigcup_{i=1}^{n+1} I_\varepsilon(f_i)\Big| - \underbrace{\Big|\bigcup_{i=1}^{n} I_\varepsilon(f_{i+1} - f_1)\Big|}_{=\left|\bigcup_{i=2}^{n+1} I_\varepsilon(f_i)\right|}.$$

invariance of Lebesgue measure

Therefore, $R_{n+1}^\varepsilon - R_n^\varepsilon \circ T \geq 2\varepsilon \cdot 1_{\{|f_i - f_1| > 2\varepsilon,\ i=2,\dots n+1\}}$,



$$\int_{\Omega_\infty}(R_{n+1}^\varepsilon - R_n^\varepsilon \circ \theta) = \int_{\Omega_\infty} R_{n+1}^\varepsilon - \int_{\Omega_\infty} R_n^\varepsilon \geq 2\varepsilon \cdot \mathbb{P}\Big(\omega\in\Omega_\infty : \underset{i=2,\dots n+1}{|f_i(\omega) - f_1(\omega)| > 2\varepsilon}\Big).$$

Since θ preserves $\mathbb{P}$,
$$\mathbb{P}\Big(\omega\in\Omega_\infty : \underset{i=2,\dots n+1}{|f_i(\omega) - f_1(\omega)| > 2\varepsilon}\Big) = \mathbb{P}\big(\omega\in\Omega_\infty : |f_i(\omega)| > 2\varepsilon,\ i=1,\dots n\big).$$

This implies that
$$0 = \overline{\lim_{n\to\infty}} \frac{1}{n}\int_{\Omega_\infty} R_n^\varepsilon \geq 2\varepsilon \cdot \mathbb{P}\big(\omega\in\Omega_\infty : |f_i(\omega)| > 2\varepsilon,\ i\geq 1\big)$$

We have shown that $\forall \varepsilon > 0$:

$$\mathbb{P}\big(\omega\in\Omega_\infty : |f_i(\omega)| > 2\varepsilon,\ i\geq 1\big) = 0.$$

Equivalently, $\mathbb{P}(\omega\in\Omega_\infty : |f_{i+p}(\omega)-f_p(\omega)|>2\varepsilon,\ i\geq 1)=0$.

Hence, $\mathbb{P}(\omega\in\Omega_\infty : \exists p,k : \forall i\geq 1 : \ |f_{i+p}(\omega)-f_p(\omega)|>\frac{1}{2^k})=0$, and

$\forall \omega\in\Omega_\infty : \forall p,k \ \ \exists i\geq 1 : \ |f_{i+p}(\omega)-f_p(\omega)|\leq \frac{1}{2^k}$.

Using this, we construct a subsequence such that

$$|f_{n_{j+1}}(\omega)-f_{n_j}(\omega)|<\frac{1}{2^j}.$$

If $\mathbb{P}(\Omega_\infty)>0$, this contradicts the assumption that

$$f_n(\omega)\xrightarrow[n\to\infty]{a.e.}\infty.$$

<u>PROOF of Thm. 1</u>: $(\Leftarrow)$

For simplicity, we give proof assuming additionally that the action of S_μ on $\Lambda^2\mathbb{C}^d$ is also strongly irreducible and contracting.

By COR. 2 (LECTURE 4),

$$\inf_n \frac{\|S_n(\omega)v\|}{\|S_n(\omega)\|}>0,\quad v\in\mathbb{C}^d\setminus\{0\},$$

$$\inf_n \frac{\|\Lambda^2 S_n(\omega)\|}{\|(\Lambda^2 S_n)w\|}>0,\quad w\in(\Lambda^2\mathbb{C}^d)\setminus\{0\},$$

$$\lim_{n\to\infty}\frac{a_2^{(n)}}{a_1^{(n)}}=\lim_{n\to\infty}\frac{\|\Lambda^2 S_n\|}{\|S_n\|^2}=0.$$

This implies that $\forall$ nonzero $v_1 \in \mathbb{C}^d$, $v_2 \in \Lambda^2 \mathbb{C}^d$:

$$\lim_{n\to\infty} \frac{\|S_n v_1\|^2}{\|(\Lambda^2 S_n) v_2\|} \overset{a.e.}{=} +\infty. \quad (\ast\ast\ast)$$

By the definition of Lyapunov exponents,

$$\lambda_1 = \lim_{n\to\infty} \frac{\log \|S_n v_1\|}{n}, \qquad (\ast\ast\ast\ast)$$

$$\lambda_1 + \lambda_2 = \lim_{n\to\infty} \frac{\log \|(\Lambda^2 S_n) v_2\|}{n}.$$

Let $P = \mathbb{P}(\mathbb{C}^d) \times \mathbb{P}(\Lambda^2 \mathbb{C}^d)$, $\nu = \nu_1 \otimes \nu_2$, where ν_i's are (unique) μ-stationary measures on the factors. We consider the action

$$GL(d, \mathbb{C}) \times P \longrightarrow P : (g, (v_1, v_2)) \longmapsto (g v_1, (\Lambda^2 g) v_2).$$

Let $\sigma(g, (v_1, v_2)) = 2 \log \frac{\|g v_1\|}{\|v_1\|} - \log \frac{\|(\Lambda^2 g) v_2\|}{\|v_2\|}$.

Then σ satisfies the cocycle property.

(in fact, σ is a combination of 2 cocycles introduced in the proof of Thm. 2).

We consider a skew-product as in the proof of Thm. 2: $\tilde{P} = \Omega \times P$, $\tilde{\nu} = \mathbb{P} \times \nu$,

$$T : \tilde{P} \to \tilde{P} : (\omega, p) \longmapsto (\theta(\omega), \omega_1 p).$$

Then $\sigma(S_n(w), x) = \sum_{k=0}^{n-1} f(T^k(w,x))$, where

$$f(w,x) = \sigma(w_1, x).$$

By $(***)$, for a.e. w,

$$\sigma(S_n(w), x) \longrightarrow +\infty.$$

Therefore, by Lemma,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k(w,x)) = \lim_{n\to\infty} \frac{1}{n} \sigma(S_n(w), x) > 0$$

for a.e. (w, x).

On the other hand, by $(\cancel{****})$,

$$\lim_{n\to\infty} \frac{1}{n} \sigma(S_n(w), x) = 2\lambda_1 - (\lambda_1 + \lambda_2) = \lambda_1 - \lambda_2.$$

Def. A subset $S \subset GL(d, \mathbb{C})$ is called *p-contracting* if there exists sequence $S_n \in S$ such that $\frac{S_n}{\|S_n\|}$ converges to rank-p matrix.

Thm 2 (Guivarc'h Raugi) Assume that S_μ is strongly irreducible and p-contracting. Then

$$\lambda_p > \lambda_{p+1}.$$

Note that if S is p-contracting, then the action of S on $\wedge^p \mathbb{C}^d$ is contracting. The proof is similar to the above proof. $\square$

Quantitative contraction properties.

For $x, y \in P = P(\mathbb{C}^d)$, $\delta(x,y) = \frac{\|x \wedge y\|}{\|x\| \cdot \|y\|}$.

Note that $\delta(x,y) = 0 \Leftrightarrow x = y$.

Thm. 3. Suppose that $\int_{GL(d,\mathbb{C})} \log^+ \|g\| \, d\mu(g) < \infty$, and S_μ is strongly irreducible and contracting.

Then:

1) $\overline{\lim_{n\to\infty}} \frac{1}{n} \log \delta(S_n(\omega)x, S_n(\omega)y) < 0$

for $x, y \in P$ and a.e. ω.

2) $\overline{\lim_{n\to\infty}} \sup_{x,y} \frac{1}{n} \int_\Omega \log \frac{\delta(S_n(\omega)x, S_n(\omega)y)}{\delta(x,y)} \, d\omega < 0$.

PROOF. We have:

$$\frac{\delta(S_n x, S_n y)}{\delta(x,y)} = \frac{\|S_n x \wedge S_n y\| \cdot \|x\| \cdot \|y\|}{\|S_n x\| \cdot \|S_n y\| \cdot \|x \wedge y\|} \leq \|\wedge^2 S_n\| \cdot \frac{\|x\| \cdot \|y\|}{\|S_n x\| \cdot \|S_n y\|}.$$

For a.e. ω, $\frac{1}{n} \log \|\wedge^2 S_n(\omega)\| \longrightarrow \lambda_1 + \lambda_2$,

$$\frac{1}{n} \log \frac{\|S_n(\omega)x\|}{\|x\|} \longrightarrow \lambda_1 \quad \text{(see Cor 2(2) Lecture 4).}$$

Therefore, $\overline{\lim\limits_{n\to\infty}} \frac{1}{n} \log \frac{\delta(S_n x, S_n y)}{\delta(x,y)} \leq (\lambda_1+\lambda_2) - 2\lambda_1 = \lambda_2 - \lambda_1 < 0.$

We note that the estimate in Cor. 2(2) is uniform over x in compact subsets of $\mathbb{C}^d \setminus \{0\}$.
Hence, by the dominated convergence thm,

$$\overline{\lim_{n\to\infty}} \sup_{x,y} \frac{1}{n} \int_\Omega \log \frac{\delta(S_n(\omega)x, S_n(\omega)y)}{\delta(x,y)}\, d\omega$$

$$\leq \lim_{n\to\infty} \frac{1}{n} \int_\Omega \log \|\wedge^2 S_n(\omega)\|\, d\omega + 2 \overline{\lim_{n\to\infty}} \sup_x \int_\Omega \frac{1}{n} \log \frac{\|x\|}{\|S_n(\omega)x\|}\, d\omega$$

$$= (\lambda_1 + \lambda_2) - 2\lambda_1 = \lambda_2 - \lambda_1 < 0. \qquad \square$$

Thm. 4. (Le Page)

Assume that $\int_{GL(d,\mathbb{C})} \|g^{\pm 1}\|^{\tau} d\mu(g) < \infty$ for some $\tau > 0$, and S_μ is strongly irreducible and contractable.

Then $\exists\, \alpha > 0,\ \rho \in (0,1)$:

$$\int_\Omega \delta(S_n(\omega)x, S_n(\omega)y)^\alpha d\omega \leq \rho^n \cdot \delta(x,y)^\alpha.$$

Proof. Let $Q_n = \sup_{x,y \in P} \int_\Omega \left(\frac{\delta(S_n(\omega)x, S_n(\omega)y)}{\delta(x,y)} \right)^\alpha d\omega.$

We need to show that $\overline{\lim\limits_{n\to\infty}} \frac{\log Q_n}{n} < 0.$

Let $\sigma(g,z) = \frac{\delta(gx, gy)}{\delta(x,y)}$ for $z = (x,y) \in P \times P$.

Then σ satisfies: $\sigma(g_1 g_2, z) = \sigma(g_1, g_2 z) \cdot \sigma(g_2, z)$.

Writing $S_{n+m}(\omega) = \underbrace{\omega_{n+m} \cdots \omega_{n+1}}_{\omega'} \cdot \underbrace{\omega_n \cdots \omega_1}_{\omega''}$,

we obtain:

$$\begin{aligned}
Q_{n+m} &= \sup_z \int \sigma(\omega', \omega'' z)^\alpha \cdot \sigma(\omega'', z)^\alpha d\omega' d\omega'' \\
&\leq \sup_z \int \left(\int \sigma(\omega', \omega'' z)^\alpha d\omega' \right) \cdot \sigma(\omega'', z)^\alpha d\omega'' \\
&\leq \left(\sup_z \int \sigma(\omega', z)^\alpha d\omega' \right) \left(\sup_z \int \sigma(\omega'', z)^\alpha d\omega'' \right) \\
&\leq Q_n \cdot Q_m.
\end{aligned}$$

Hence, $\overline{\lim\limits_{n\to\infty}} \frac{\log Q_n}{n} = \inf \frac{\log Q_n}{n}.$

It is sufficient to show that $Q_n < 1$ for some n.
We use the estimate: $e^x \leq 1 + x + \frac{x^2}{2}\exp(|x|)$. Then:

$$Q_n \leq 1 + \alpha \cdot \underbrace{\left(\sup_z \int_\Omega \log \sigma(S_n(\omega), z)\, d\omega\right)}_{A_n}$$

$$+ \frac{\alpha^2}{2} \underbrace{\sup_z \int_\Omega \left(\log \sigma(S_n(\omega), z)\right)^2 \cdot \sigma(S_n(\omega), z)^\alpha\, d\omega}_{B_n}.$$

By Thm 3 (2),

$$\overline{\lim_{n\to\infty}} \frac{A_n}{n} = \overline{\lim_{n\to\infty}} \sup_{x,y} \frac{1}{n} \int_\Omega \frac{\delta(S_n(\omega)x, S_n(\omega)y)}{\delta(x,y)}\, d\omega < 0.$$

By the Cauchy-Schwartz inequality,

$$B_n \leq \sup_z \left(\int_\Omega \log^4 \sigma(S_n(\omega), z) d\omega\right)^{1/2} \cdot \left(\int_\Omega \sigma(S_n(\omega), z)^{2\alpha} d\omega\right)^{1/2}.$$

We have:

$$\sigma(g, z) = \frac{\|gx \wedge gy\| \cdot \|x\| \cdot \|y\|}{\|gx\| \cdot \|gy\| \cdot \|x \wedge y\|} \leq \|\wedge^2 g\| \cdot \|g^{-1}\|^2 \leq \text{const} \cdot \|g\|^2 \cdot \|g^{-1}\|^2.$$

Hence, $\int_\Omega \sigma(S_n(\omega), z)^\beta d\omega \leq \text{const} \cdot \left(\int_{GL(d,\mathbb{C})} \|g\|^{2\beta} \cdot \|g^{-1}\|^{2\beta} d\mu(g)\right)^n$

is finite for sufficiently small $\beta > 0$.
This implies that $B_n < \infty$ for sufficiently small α.

Finally, $Q_n \leqslant 1 + \alpha \cdot \underbrace{A_n}_{<0} + \frac{\alpha^2}{2} \cdot \underbrace{B_n}_{<\infty}$.

Taking α sufficiently small gives $Q_n < 1$.