Lecture 6: Contracting elements and Zariski topology.

Def. We say that an element $g \in GL(d, \mathbb{C})$ is contracting (proximal) if it has unique eigenvalue of maximal modulus.

Question. How typical for g to be contracting?

Zariski topology on $GL_d(\mathbb{R})$:

For $I \subset \mathbb{R}[x_{11}, \dots x_{dd}]$, we set
$$V(I) = \{ g \in GL_d(\mathbb{R}) : f(g) = 0 \ \forall f \in I \}.$$

ex. 1) $\bigcap_\alpha V(I_\alpha) = V(\bigcup_\alpha I_\alpha)$,

2) $V(I_1) \cup V(I_2) = V(I_1 \cap I_2)$.

Def The Zariski topology is topology whose closed sets are $V(I)$, $I \subset \mathbb{R}[x_{11}, \dots x_{dd}]$.

Rmk. Zariski topology is much coarser than the usual topology. For example, every Zariski closed subgroup of $SL_2(\mathbb{R})$ is either compact or solvable group.

Compactness property.

If $\{U_\alpha\}$ is an family of open sets,

then $\bigcup_\alpha U_\alpha = \bigcup_{i=1}^{k} U_{\alpha_i}$.

(This follows from ideals in $\mathbb{R}[x_{11}, \dots x_{dd}]$ being finitely generated.)

$$\boxed{\text{Thm. (Goldsheid–Margulis) Every Zariski-dense subgroup of } SL_d(\mathbb{R}) \text{ contains contracting elements.}}$$

Lem. 1: Let Γ be a subsemigroup of $GL(d, \mathbb{C})$ such that for $\gamma \in \Gamma$, all eigenvalues λ satisfy $|\lambda| \leq 1$, and Γ acts irreducibly on $\mathbb{C}^d$.

Then Γ is relatively compact.

In the proof, we use:

Thm (Burnside) Let $G \subset GL(d, \mathbb{C})$ be a subsemigroup acting irreducibly on $\mathbb{C}^d$. Then

$$\operatorname{span}(G) = M(d, \mathbb{C}).$$

Proof of lemma. The map $B(x,y) = \mathrm{Tr}(xy)$ defines a nondegenerate bilinear form on $M(d,\mathbb{C})$.
By Burnside's Thm, $\exists \{\gamma_i\}_{i=1}^{d^2} \subset \Gamma$ – a basis of $M(d,\mathbb{C})$.
Let $\{\gamma_i^*\}$ be a dual basis, i.e., $B(\gamma_i, \gamma_j^*) = \delta_{ij}$.
Then $\gamma = \sum_i B(\gamma, \gamma_i)\gamma_i^*$ where $B(\gamma,\gamma_i) = \mathrm{Tr}(\gamma\gamma_i)$ are uniformly bounded by our assumption.
Hence, Γ is bounded. $\square$

Lem. 2 Compact subsemigroup $G \subset GL(d,\mathbb{C})$ is a group.

Proof. For $g \in G$, $\exists$ sequence $\{n_i\}$: $n_{i+1} > n_i$: $g^{n_i} \to g_0$.
Then $g^{n_{i+1}-n_i-1} \to g^{-1} \in G$. $\square$

Lem. 3. Let Γ be a subsemigroup of $GL(d,\mathbb{R})$ such that $\forall \gamma \in \Gamma$ has eigenvalues of the same modulus. Then the Zariski closure has the same property.

Proof. Let $\Gamma_0 = \{ \frac{\gamma}{|\det(\gamma)|^{1/d}} : \gamma \in \Gamma \}$.
Let $V_0 \subset \dots \subset V_n$ be a maximal flag of subspaces invariant under Γ (equivalently, Γ_0).
Then Γ_0 acts irreducibly V_{i+1}/V_i and

eigenvalues λ of $\gamma \in \Gamma_0$ satisfy $|\lambda| \leq 1$.

By Lem. 2, $\Gamma_0 \hookrightarrow GL(V_{i+1}/V_i)$ is bounded.

We claim that the image is contained in the orthogonal group $O(Q) = \{g : Q(gv) = Q(v)\}$ for a positive-definite quadratic form.

Let $G \subset GL(V_{i+1}/V_i)$ be (Euclidean) closure of Γ_0. By Lem. 2, G is a group.

<u>Fact:</u> If G is compact group, then $\exists$ G-inv. probability measure m on G.

Let Q_0 be any positive definite quad. form, and $Q(v) = \int_G Q(g \cdot v)\, dm(g)$. Then Q is G-invariant and positive-definite.

We have shown that, in suitable basis,

$$\Gamma \subset \mathbb{R}^\times \cdot \begin{pmatrix} O(Q_n) & & * \\ & \ddots & \\ 0 & & O(Q_1) \end{pmatrix}$$

The same is also true for $\overline{\Gamma}$.

This implies lemma. $\square$

ex. $SL(d,\mathbb{R}) \neq X_1 \cup X_2$ where X_i's are proper Zariski closed subsets.

Hint: Use that $f(x) = \det(x) - 1$ is irreducible polynomial

$\Downarrow$

$\forall$ open $\mathcal{U}_1, \mathcal{U}_2 \neq \emptyset$ in $SL(d,\mathbb{R})$: $\mathcal{U}_1 \cap \mathcal{U}_2 \neq \emptyset$

$\Downarrow$ $\forall$ open $\mathcal{U} \neq \emptyset$ in $SL(d,\mathbb{R})$ is dense.

Proof of Theorem. For $g \in SL(d,\mathbb{R})$,

$m_g = \#$(of eigenvalues λ of g with $|\lambda|$ maximal).

Let $m = \min\{m_\gamma : \gamma \in \Gamma\}$. Suppose that $m > 1$.

By Lem. 3, $m < d$.

Let $\Gamma_0 = \{\gamma \in \Gamma : m_\gamma = m\}$ and $\mathcal{R} = \left\{g \in SL(d,\mathbb{R}) : \begin{matrix}\text{all eigenvalues} \\ \text{of } g \text{ are distinct}\end{matrix}\right\}$.

Claim. $\Gamma_0 \cap \mathcal{R} \neq \emptyset$

Take $\gamma \in \Gamma$: $m_\gamma = m$. Then $(\Lambda^m \gamma)$ is contracting,

and $\dfrac{\Lambda^m \gamma^n}{\|\Lambda^m \gamma^n\|} \xrightarrow[n\to\infty]{} P$ (after passing to subsequence),

where P is a rank-1 linear map of $\Lambda^m \mathbb{R}^d$.

Let $\mathcal{U} = \{g \in SL(d,\mathbb{R}) : P(\Lambda^m g)P \neq 0\}$.

This is a nonempty open subset of $SL(d,\mathbb{R})$.

$\mathcal{R}$ is also a nonempty open subset of $SL(d,\mathbb{R})$.

Hence, $\Gamma_1 = \Gamma \cap \mathcal{U} \cap \mathcal{R}$ is dense in $SL(d,\mathbb{R})$.

$\forall x \in \Gamma_1$: $\dfrac{\Lambda^m(\gamma^n x)}{\|\Lambda^m \gamma^n\|} \xrightarrow[n\to\infty]{} P.(\Lambda^m x)$ – Rank-1 map.
$$\underset{0}{\#}$$

This implies that sufficiently large n, $\Lambda^m \gamma^n x$ is contracting, and $m_{\gamma^n x} = m \Rightarrow \gamma^n x \in \Gamma_0$.

By the compactness property,
$$\bigcup_{n\geq 1} \gamma^n \mathcal{R} x^{-1} = \bigcup_{n=1}^{r} \gamma^n \mathcal{R} x^{-1}.$$

Hence, $\forall n \; \exists i \leq r$: $\gamma^{n-i} \in \mathcal{R} x^{-1} \Rightarrow \gamma^{n-i} x \in \mathcal{R}$.

This implies the claim.

Take $\gamma \in \Gamma_0 \cap \mathcal{R}$. Since $\gamma \in \mathcal{R}$ and $m_\gamma = m$, after passing to subsequence, $\lambda^{-n} \cdot \gamma^n \to P$ where P is rank-m matrix. We set $V = \operatorname{Im}(P)$, and $\mathcal{U} = \{g : PgP \text{ is non-degenerate on } V\}$. Then $\mathcal{U}$ is nonempty Zariski-open subset, and $\mathcal{U} \cap \Gamma$ is Zariski dense in $SL(d,\mathbb{R})$.

Let H be the subsemigroup of $GL(V)$ generated by $P\gamma P$, $\gamma \in \mathcal{U} \cap \Gamma$. Then $\overline{H} \supset P.\,SL_d(\mathbb{R}) \cdot P$.

By Lem. 3, $\exists\, h \in H$: $h = (P\gamma_1 P) \dots (P\gamma_\ell P)$ such that eigenvalues of h have at least two different moduli. Let $\delta_n = (\gamma^n \gamma_1 \gamma^n) \dots (\gamma^n \gamma_\ell \cdot \gamma^n)$.

Then $\lambda^{-2\ell n} \cdot \delta_n \to h$, and sufficiently large n,

$$m_{\delta_n} \leq m_h < m,$$ which is a contradiction.