# Lecture 7: Amenability.

$G$ = closed subgroup of $GL(d, \mathbb{C})$.

**Def.** $G$ is called amenable if for every continuous action of $G$ on compact space $X$, $\exists$ $G$-inv. probability measure on $X$.

**ex** 1) If $G$ is noncompact and totally irreducible, then by Furstenberg's lemma, $P(\mathbb{C}^d)$ has no $G$-inv. prob. measures, and $G$ is not amenable.

2) If $G$ is compact, then it is known that $\exists$ $G$-inv. measure $m$ on $G$ (Haar measure). For action of $G$ on $X$, we set

$$\int_X f\, d\mu = \int_G f(gx_0)\, dm(g).$$

Then $m$ is $G$-inv.
Hence, $G$ is compact.

## 1) Fix-point properties.

**Thm.** $G$-amenable $\Leftrightarrow$ $\forall$ continuous linear action of $G$ on a locally convex top. vector space $V$ $\forall$ convex compact $G$-inv. $C \subset V$, $C \neq \emptyset$, $\exists$ fixed point $x \in C$.

PROOF. $\Leftarrow$ Given an action $G$ on compact space $X$, let $\mathcal{P}(X) = \{\text{prob. measures}\}$ equipped with weak* topology. Then $\mathcal{P}(X)$ is compact, and contains a fixed point, which is a $G$-inv. measure.

$\Rightarrow$ We introduce barycenter map $b : \mathcal{P}(C) \to C$. If $\mu = \sum_i \alpha_i \delta_{c_i}$, $c_i \in C$, $\alpha_i \geq 0$, $\sum_i \alpha_i = 1$, then $b(\mu) = \sum_i \alpha_i \cdot c_i$. Note that

$$b(g \cdot \mu) = g \cdot b(\mu). \qquad (*)$$

One shows that $b$ extends continuously to $\mathcal{P}(C)$, and $(*)$ holds. Then if $\mu \in \mathcal{P}(C)$ is $G$-inv., then $b(\mu) \in C$ is fixed by $G$. $\square$

COR. If $N$ is a closed normal subgroup of $G$ such that $N$ and $G/N$ is amenable. Then $G$ is amenable.

Thm. Abelian groups are amenable.

PROOF. Consider a linear action of $G$ on nonempty convex compact set $C$. For $g \in G$, let $A_n(g) = \frac{1}{n} \sum_{i=0}^{n-1} g^i$. Let $S$ be the semigroup of linear maps generated by $A_n(g)$, $n \geq 1$, $g \in G$.

We claim that $C_\infty = \bigcap_{s \in S} s(C) \neq \emptyset$. Since $s(C)$ is compact, it is sufficient to show that $\bigcap_{i=1}^{n} s_i(C) \neq \emptyset$. Let $s = s_1 \dots s_n$.

Since $S$ is abelian, $s(C) \subset s_i(C)$ for all $i$.

This implies the claim.

Take $c \in C_\infty$. Then $\forall n \geq 1 : \forall g \in G : \quad c = A_n(g) \cdot d$ for some $d \in C$, and

$$g \cdot c - c = \frac{g^{n+1} d - d}{n} \xrightarrow[n \to \infty]{} 0.$$

Hence, $c$ is fixed by $G$. $\square$

<u>Cor.</u> Solvable groups are amenable.

2) Invariant means.

$V$ = closed subspace of $L^\infty(G)$, $1 \in V$.

<u>Def.</u> A <u>mean</u> $M$ on $V$ is a linear map such that $M(1) = 1$, $\forall \varphi \in V : \varphi \geq 0 \Rightarrow M(\varphi) \geq 0$.

<u>Rmk.</u> We have $-\|\varphi\|_\infty \leq M(\varphi) \leq \|\varphi\|_\infty$, so that $|M(\varphi)| \leq \|\varphi\|_\infty$, and $M$ is automatically continuous.

**Def** $M$ is G-inv. if $M(g.\varphi) = M(\varphi)$, $g \in G$.

The space of uniformly continuos bounded functions:

$$UCB(G) = \{\varphi \in L^\infty(G) : \lim_{y\to e} \sup_{x\in G} |\varphi(y.x) - \varphi(x)| = 0\}$$

[**Thm** $G$ is amenable $\Leftrightarrow$ $\exists$ G-inv. mean on $V = UCB(G)$.]

**Proof.** $\Rightarrow$ We equipp the dual space $V^*$ with weak* topology (i.e., weakest topology for which $\alpha \mapsto \alpha(v)$, $v \in V$ is continuous).

Then $\mathcal{M}$ (= the space of means) is weak* closed, bounded subset. Hence, $\mathcal{M}$ is (weak*) compact.

Also, $\mathcal{M} \neq \emptyset$, since $\varphi \mapsto \varphi(e)$ is a mean.

Consider action of $G$ on $\mathcal{M}$: $(g.m)(\varphi) = m(g.\varphi)$.

Since $G$ is amenable, $\exists$ fixed point which is a G-inv. mean.

$\Leftarrow$ Consider linear action of $G$ on nonempty convex compact set $C$. Fix $c_0 \in C$. For $f \in C(C)$, define $\hat{f}(g) = f(g\,c_0)$. Then $\hat{f} \in UCB(G)$.

Let $M$ be a G-inv. mean on $UCB(G)$.

We define prob. measure on $C$ by

$$\int_C f\, d\mu = M(\hat{f}), \quad f \in C(C).$$

Since $\widehat{g\cdot f} = g\cdot\hat{f}$, $\mu$ is $G$-inv., and the barycenter $b(\mu) \in C$ is fixed by $G$. $\square$

3) Absence of spectral gap:

Fact: $G$ is equipped with left $G$-inv. measure (Haar measure).

$$\mathcal{H} = L^2(G) = \{ f : G \to \mathbb{C} : \int_G |f(g)|^2 dg < \infty \}.$$

For $g \in G$, $\pi(g) : \mathcal{H} \longrightarrow \mathcal{H}$

$$f \longmapsto f(g^{-1}x)$$

Fix $\varphi : G \to \mathbb{R}^+$, $\int_G \varphi = 1$ such that $G = \overline{\langle \mathrm{supp}(\varphi) \rangle}$, $\varphi(g^{-1}) = \varphi(g)$.

Let $\mu(g) = \varphi(g)\, dg$, and

$$\pi(\mu) = \int_G \pi(g)\, d\mu(g) : \mathcal{H} \to \mathcal{H}.$$

Thm. $G$ is amenable $\Leftrightarrow$ $\|\pi(\mu)\| = 1$.

Proof. $\Leftarrow$ Since $f$ is symmetric, $\pi(\mu)^* = \pi(\mu)$.

Since $\|\pi(\mu)\| = \sup_{\|f\|_2=1} \langle \pi(\mu)f, f\rangle$, $\exists f_n \in \mathcal{H} : \|f_n\|_2 = 1$, $\langle \pi(\mu)f_n, f_n\rangle \to 1$.

We argue as in Lecture 2 (proof of Prop.) to show that $\exists \psi_n \geq 0 : \|\psi_n\|_1 = 1$ such that

$$\|\pi(g)\psi_n - \psi_n\|_1 \xrightarrow[n\to\infty]{} 0 \text{ for } \mu\text{-a.e. } g \in G.$$

We define a sequence of means on $UCB(G)$:

$$M_n(\varphi) = \int_G \psi_n(g)\varphi(g)\,dg.$$

We have $\left|M_n(g_0\varphi) - M_n(\varphi)\right| = \left| \int_G \psi_n(g_0^{-1}g)\varphi(g)\,dg \right.$

$$\left. - \int_G \psi_n(g)\varphi(g)\,dg \right| \leq \int_G |\pi(g_0)\psi_n(g) - \psi_n(g)| \cdot |\varphi(g)|\,dg$$

$$\leq \|\pi(g_0)\psi_n - \psi_n\|_1 \cdot \|\varphi\|_\infty \to 0.$$

By weak* compactness, $M_{n_i} \to M$, and $M$ is a $G$-inv. mean.

$\Rightarrow$ (skipped) $\square$

Rmk. The proof of $\Leftarrow$ works for any symmetric prob. measure.

# Amenability and growth of random walks.

Let $\mu$ be a symmetric prob. measure on $G$ such that $G = \langle supp(\mu) \rangle$.

Let $\Omega = G^{\mathbb{N}}$ with $\mathbb{P} = \mu^{\otimes \mathbb{N}}$.

For $\omega \in \Omega$: $S_n(\omega) = \omega_n \dots \omega_1$.

Let $d$ be a left invariant metric on $G$.

We assume that $\int_G d(g,e)\,d\mu(g) < \infty$.

__Lem.__ For a.e. $\omega \in \Omega$, $\lim_{n\to\infty} \frac{1}{n} d(S_n(\omega), e)$ exists and constant.

__Proof.__ $d(S_{n+m}(\omega), e) = d(S_n(\Theta^m \omega) S_m(\omega), e)$

$$= d(S_m(\omega), S_n(\Theta^m\omega)^{-1}) \leq d(S_m(\omega), e) + d(e, S_n(\Theta^m\omega)^{-1})$$

$$= d(S_m(\omega), e) + d(S_n(\Theta^m\omega), e).$$

Now lemma follows from the Subadditive Ergodic Thm. $\square$

__Def.__ $\lambda(\mu) = \lim_{n\to\infty} \frac{1}{n} d(S_n, e)$ is called the __escape rate__ of the random walk.

Let $\delta_G = \overline{\lim_{R\to\infty}} \frac{1}{R} \log \mathrm{Vol}\{g\in G : d(g,e)<R\}$.

**Thm** (Furman–Shalom) Assume that $\delta<\infty$ and the group $G$ is not amenable. Then

$$\lambda(\mu) \geq \frac{2}{\delta_G} \log\left(\|\pi(\mu)\|^{-1}\right) > 0.$$

**Proof.** Let $K$ be a compact subset of $G$ with $\mathrm{Vol}(K)>0$. Let

$$f_1 = \frac{1}{\mathrm{Vol}(K)} \cdot \chi_K \quad \text{and} \quad f_2(g) = e^{-\frac{\delta}{2} d(g,e)}.$$

with $\delta > \delta_G$. Note that $f_1 \in L^2(G)$, and

$$\int_G |f_2|^2 = \int_G e^{-\delta d(g,e)} dg \leq \sum_n e^{-\delta n} \cdot \underbrace{\mathrm{Vol}(n \leq d(g,e) < n+1)}_{\leq e^{-(\delta_G-\varepsilon)(n+1)},\ \varepsilon>0} < \infty.$$

By Jensen inequality,

$$-\log\langle \pi(g_0) f_2, f_1\rangle = -\log\left(\int_G e^{-\frac{\delta}{2} d(g,g_0)} f_1(g)\,dg\right)$$

$$\leq \int_G \frac{\delta}{2} d(g,g_0) \cdot f_1(g)\,dg \leq \int_G \frac{\delta}{2}\left(d(g_0,e)+d(g,e)\right) f_1(g)\,dg$$

$$\leq \frac{\delta}{2} d(g_0,e) + \underbrace{\mathrm{const}}_{\text{depends only on } K}.$$

Therefore, it remains to show that for a.e. $\omega$,

$$\underline{\lim}_{n\to\infty} -\frac{1}{n}\log \langle \pi(S_n(\omega))f_2, f_1\rangle \geq \log\left(\|\pi(\mu)\|^{-1}\right). \quad (*)$$

Let $\varepsilon > 0$ and $h_\varepsilon(\omega) = \sum_{n\geq 0} (\|\pi(\mu)\|+\varepsilon)^{-n} \langle \pi(S_n(\omega))f_2, f_1\rangle$

$$\int_\Omega h_\varepsilon(\omega)\,d\omega = \sum_{n\geq 0} (\|\pi(\mu)\|+\varepsilon)^{-n} \cdot \int_{G^n} \langle \pi(\omega_n \ldots \omega_1) f_2, f_1\rangle \,d\mu(\omega_1)\ldots d\mu(\omega_n)$$

$$= \sum_{n\geq 0} (\|\pi(\mu)\|+\varepsilon)^{-n} \cdot \langle \underbrace{\pi(\mu^{*n})}_{\pi(\mu)^n} f_2, f_1\rangle$$

$$\leq \sum_{n\geq 0} \left(\frac{\|\pi(\mu)\|}{\|\pi(\mu)\|+\varepsilon}\right)^n \cdot \|f_1\|\cdot\|f_2\| < \infty.$$

Hence, $h_\varepsilon \overset{a.e.}{<} \infty$, and for a.e. $\omega$,

$$\overline{\lim}_{n\to\infty} \frac{1}{n}\log\langle \pi(S_n(\omega))f_2, f_1\rangle \leq \log(\|\pi(\mu)\|+\varepsilon),$$

for every $\varepsilon > 0$. This implies $(*)$.

<u>Cor.</u> 1) If $G$ has subexponential volume growth (i.e., $\delta_G = 0$), then $G$ is amenable.

2) If $G$ is not amenable, then $\lambda(\mu) > 0$.