Lecture 8: Boundaries and random walks.

μ = probability measure on $GL(d,\mathbb{C})$,

$G = \overline{\langle supp(\mu) \rangle}$,

$\Omega = G^{\mathbb{N}}$ with prob. measure $\mathbb{P}$.

Question: 1) Describe limit behaviour of $\omega_{\le n} = \omega_1 \dots \omega_n$ for a.e. $\omega \in \Omega$.

2) "Compactify" $G \subset \overline{G}$ so that $\omega_{\le n}$ converges in $\overline{G}$.

Def. A compact prob. space (P,ν) is called μ-boundary if G acts on P so that

1) ν is μ-stationary (i.e., $\mu * \nu = \nu$)

2) for a.e. $\omega \in \Omega$, $\omega_{\le n} \nu \xrightarrow[n\to\infty]{} \delta_{x(\omega)}$ - Dirac measure

examples:

1) If S_μ is totally irreducible and contracting, $\exists!$ stationary measure ν on $\mathbb{P}(\mathbb{C}^d)$ and $(\mathbb{P}(\mathbb{C}^d), \nu)$ is a μ-boundary.

2) If S_μ is totally irreducible and contracting on $\mathbb{P}(\Lambda^k \mathbb{C}^d)$, then $(\mathbb{P}(\Lambda^k\mathbb{C}^d), \nu_k)$ is a μ-boundary.

Question: Is there a "universal" μ-boundary?

Let (P, ν) be a μ-boundary.

We define a map $\Psi : G \cup P \longrightarrow \mathcal{P}(P)$

space of prob. measures on P.

$$\Psi(g) = g \cdot \nu, \quad g \in G,$$
$$\Psi(p) = \delta_p, \quad p \in P.$$

Then $G/\mathrm{Stab}(\nu) \hookrightarrow \mathrm{Im}(\Psi)$ defines a compactification of $G/\mathrm{Stab}(\nu)$, and in this compactification random products converge.

Thm. Assume that G contains a closed amenable subgroup H such that G/H is compact.
Let (P, ν) be a μ-boundary.
Then $\exists$ closed subgroup $H' \supset H$ such that $\mathrm{supp}(\nu) \simeq G/H'$ as a G-space.

<u>Proof.</u> Since H is amenable, $\exists$ H-inv. probability measure on P. Since P is compact, passing to a subsequence, $\frac{1}{n_i}\sum_{j=1}^{n_i}\mu^{*j}*m \xrightarrow[i\to\infty]{} \nu$ for some prob. measure ν. It is easy to check that $\mu*\nu=\nu$.

Let $\pi: G\to G/H$ denote the canonical projection. Since G/H is compact, passing to a subsequence $\pi\left(\frac{1}{n_i}\sum_{j=1}^{n_i}\mu^{*n_i}\right)\longrightarrow \rho$ for some prob. measure ρ.

Pick a prob. measure λ on G such that $\pi(\lambda)=\rho$. Since m is H-inv.,

$$\frac{1}{n_i}\sum_{j=1}^{n_i}\mu^{*n_i}*m \xrightarrow[i\to\infty]{} \lambda*m=\nu.$$

Since (P,ν) is a μ-boundary, $\exists g_n\in G:\ g_n\cdot\nu\longrightarrow\delta_z$. Let $\{V_i\}$ be a basis of neighbourhoods of z. Then for sufficiently large n,

$$(g_n\nu)(V_i)=\int_G (g\cdot m)(V_i)\,d(g_n\cdot\lambda)(g)>1-\frac{1}{i}.$$

This implies that $\exists \ell_i \in G: (\ell_i m)(V_i) > 1 - \frac{1}{i}$.

This yields $\ell_i \cdot m \xrightarrow[i\to\infty]{} \delta_z$.

We write $G = K \cdot H$ where K is a compact subset of G. Then $\ell_i = k_i \cdot h_i$, $k_i \in K$, $h_i \in H$.

Passing to a subsequence, $k_i \to k$.

Then $\ell_i m = k_i m \longrightarrow km = \delta_z \Rightarrow m = \delta_{k^{-1}z}$.

Let $H' = \mathrm{Stab}_G(k^{-1}z) \supset H$.

Clearly, $\mathrm{supp}(\nu) = G \cdot (k^{-1}z) \simeq G/H'$. $\square$

<u>Cor.</u> Let μ be a <u>prob. measure</u> on $SL(d,\mathbb{R})$ such that $\langle \mathrm{supp}(\mu) \rangle = SL(d,\mathbb{R})$.

Let (P,ν) be a μ-boundary.

Then $\exists$ a space of flags $\mathcal{F}_\tau$:

$$\mathrm{supp}(\nu) \simeq \mathcal{F}_\tau \text{ as } SL(d,\mathbb{R})\text{-spaces.}$$

<u>Proof.</u> We have Iwasawa decomposition:

$$SL(d,\mathbb{R}) = K \cdot H$$

where $K = SO(d)$ and $H = \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix}$

(this follows from Gram-Schmidt orthogonalization).

By Thm, $\mathrm{supp}(\nu) \simeq SL(d,\mathbb{R})/H'$ where H' is closed subgroup $H' \supset H$.

Such $H' = \begin{pmatrix} * & & * \\ & * & \\ 0 & & * \end{pmatrix}$ and

$SL(d,\mathbb{R})/H' \simeq \mathcal{F}_\tau$ for some τ. $\square$