

Lecture 8: Boundaries and random walks.

μ = probability measure on $GL(d, \mathbb{C})$,

$G = \overline{\text{supp}(\mu)}$,

$\Omega = G^N$ with prob. measure P .

Question: 1) Describe limit behaviour of
 $w_{\leq n} = w_1 \dots w_n$ for a.e. $w \in \Omega$.

2) "Compactify" $G \subset \overline{G}$ so that
 $w_{\leq n}$ converges in \overline{G} .

Def. A compact prob. space (P, ν) is called
 μ -boundary if G acts on P so that

1) ν is μ -stationary (i.e., $\mu * \nu = \nu$)

2) for a.e. $w \in \Omega$, $w_{\leq n} \nu \xrightarrow{n \rightarrow \infty} \delta_{x(w)}$ - Dirac measure

examples:

1) If S_μ is totally irreducible and contracting, $\exists!$ stationary measure ν on $P(\mathbb{C}^d)$ and $(P(\mathbb{C}^d), \nu)$ is a μ -boundary.

2) If S_μ is totally irreducible and contracting on $P(\Lambda^\kappa \mathbb{C}^d)$, then $(P(\Lambda^\kappa \mathbb{C}^d), \kappa)$ is a μ -boundary.

Question: Is there a "universal" μ -boundary?

Let (P, ν) be a μ -boundary.

We define a map $\Psi: G \times P \rightarrow \mathcal{P}(P)$
↪ space of prob. measures on P .

$$\begin{aligned}\Psi(g) &= g \cdot \nu, \quad g \in G, \\ \Psi(P) &= \delta_P, \quad P \in \mathcal{P}.\end{aligned}$$

Then $G/\text{stab}(\nu) \hookrightarrow \text{Im}(\Psi)$ defines a compactification of $G/\text{stab}(\nu)$, and in this compactification random products converge.

Thm. Assume that G contains a closed amenable subgroup H such that G/H is compact.
 Let (P, ν) be a μ -boundary.
 Then \exists closed subgroup $H' \trianglelefteq H$ such that $\text{supp}(\nu) \cong G/H'$ as a G -space.

Proof. Since H is amenable, $\exists H$ -inv. probability measure on P . Since P is compact, passing to a subsequence, $\frac{1}{n_i} \sum_{j=1}^{n_i} \mu^{*j} * m \xrightarrow[i \rightarrow \infty]{} \nu$ for some prob. measure ν . It is easy to check that $\mu * \nu = \nu$.

Let $\pi: G \rightarrow G/H$ denote the canonical projection. Since G/H is compact, passing to a subsequence $\pi\left(\frac{1}{n_i} \sum_{j=1}^{n_i} \mu^{*n_i}\right) \xrightarrow{} \rho$ for some prob. measure ρ . Pick a prob. measure λ on G such that $\pi(\lambda) = \rho$. Since m is H -inv.,

$$\frac{1}{n_i} \sum_{j=1}^{n_i} \mu^{*n_i} * m \xrightarrow[i \rightarrow \infty]{} \lambda * m = \nu.$$

Since (P, ν) is a μ -boundary, $\exists g_n \in G: g_n \cdot \nu \xrightarrow{} \delta_z$. Let $\{V_i\}$ be a basis of neighbourhoods of z . Then

for sufficiently large n ,

$$(g_n \nu)(V_i) = \int_G (g \cdot m)(V_i) d(g_n \cdot \lambda)(g) > 1 - \frac{1}{i}.$$

This implies that $\exists i \in \mathbb{G} : (\lim)(v_i) > 1 - \frac{1}{i}$.

This yields $\lim_{i \rightarrow \infty} l_i \cdot m \rightarrow \delta_z$.

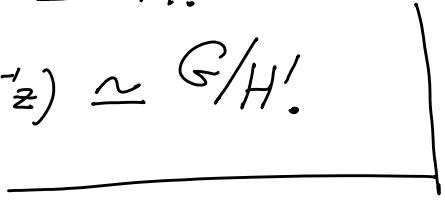
We write $G = K \cdot H$ where K is a compact subset of G . Then $l_i = k_i \cdot h_i$, $k_i \in K$, $h_i \in H$.

Passing to a subsequence, $k_i \rightarrow k$.

Then $\lim_{i \rightarrow \infty} k_i \cdot m \rightarrow km = \delta_z \Rightarrow m = \delta_{k^{-1}z}$.

Let $H' = \text{Stab}_G(k^{-1}z) \supset H$.

Clearly, $\text{supp}(\nu) = G \cdot (k^{-1}z) \cong G/H'$.



Cor. Let μ be a prob. measure on $SL(d, \mathbb{R})$ such that $\overline{\text{supp}(\mu)} = SL(d, \mathbb{R})$.

Let (P, ν) be a μ -boundary.

Then \exists a space of flags \mathcal{F}_ν :

$\text{supp}(\nu) \cong \mathcal{F}_\nu$ as $SL(d, \mathbb{R})$ -spaces.

Proof. We have Iwasawa decomposition:

$$SL(d, \mathbb{R}) = K \cdot H$$

where $K = SO(d)$ and $H = \begin{pmatrix} * & * & \\ 0 & \ddots & * \end{pmatrix}$

(this follows from Gram-Schmidt orthogonalization).

By Thm, $\text{supp}(\nu) \cong \text{SL}(d, \mathbb{R}) / H'$

where H' is closed subgroup $H' \supset H$.

Such $H' = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ and

$\text{SL}(d, \mathbb{R}) / H' \cong \mathcal{E}$ for some \mathcal{C} .