Lecture 9: Selected further developments.

G = a (locally compact) group,

μ = prob. measure on G with $\overline{\langle \mathrm{supp}(\mu) \rangle} = G$.

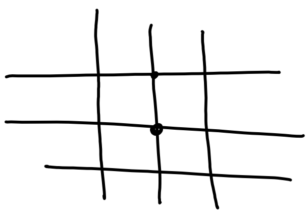
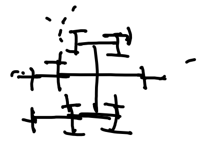
Random walk: $S_n = \omega_n \dots \omega_1$ where ω_i's independent with distribution μ.

Algebraic properties of groups	$\overset{?}{\longleftrightarrow}$	Asymptotic properties of random walks
• commutativity		• recurrence
• growth		• boundary
• amenability		• stationary measure on the boundary

For simplicity, we assume that G is finitely generated.

1) Kesten Problem:

Def. Random walk S_n is called recurrent if almost surely, $S_n = e$ for inf. many n.

Kesten Problem: Which groups support recurrent random walks?

Thm (Varopoulos)
G supports a recurrent random walk $\Leftrightarrow$ G is a finite extension of $\mathbb{Z}^d$, $d=0,1,2$.

2) Growth & Boundary:

Def. Fix a finite generating set $S \subset G$.
Let $g(n) = |S^n|$.

1) G has polynomial growth if $g(n) \leq c \cdot n^d$
2) G has exponential growth if $g(n) \geq c \cdot e^{\alpha n}$.
3) Otherwise, G has intermediate growth.

Thm. If G has polynomial growth, then every μ-boundary is trivial.

Thm (Avez) If G has a finitely-supported μ with nontrivial boundary, then G has exponential growth.

Examples (Kaimanovich–Vershik, Bartholdi–Erschler)
$\exists G$ with exponential growth such that for all finitely supported μ, the μ-boundary is trivial.

Conj (Kaimanovich–Vershik) For every G with exponential growth, $\exists$ symmetric μ on G with nontrivial μ-boundary.

example (Erschler)
$\exists G$ of intermediate growth with nontrivial μ-boundary for symmetric (infinitely supported) μ.

Thm (Kaimanovich–Vershik)
G is amenable $\Leftrightarrow$ $\exists \mu$ with $\mathrm{supp}(\mu) = G$ such that μ-boundary is trivial.

(This is not true if μ is finitely supported.)

3) Properties of stationary measures

μ = prob. measure on $SL(d,\mathbb{R})$

$\mathcal{F}$ = the space of flags

ν = μ-stationary measure on $\mathcal{F}$.

$$\boxed{\mu * \nu = \nu}$$

Thm (Furstenberg) If μ is absolutely continuous, then ν is also absolutely continuous.

Thm. (Furstenberg) $\exists \mu$: $supp(\mu) \subset SL(d,\mathbb{Z})$ with ν absolutely continuous.

Conj (Kaimanovich, Le Prince)
For any finitely supported μ on $SL(d,\mathbb{R})$, the corresponding ν is singular.

Rmk. Barany - Pollicott - Simon gave a counterexample.
This conjecture might still hold if
- μ is symmetric, or
- $\langle supp(\mu) \rangle$ is Zariski dense in $Sld(\mathbb{R})$ with $d \geq 3$.

Thm (Kaimanovich – Le Prince)

For any Zariski dense fin. generated subgroup $G \subset SL(d, \mathbb{R})$, $\exists \mu$ with finite support $\langle supp(\mu) \rangle = G$ such that the corresponding ν is singular.