A proof of the Willmore Conjecture II

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Plan

1. Given $\Sigma \subset S^3$ define the canonical family

$$C : B^4 \times [-\pi, \pi] \to \mathcal{Z}_2(S^3).$$

2. Show that

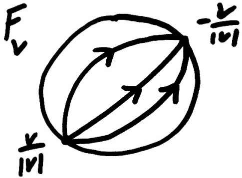
$$area(C(v,t)) \leq \mathcal{W}(\Sigma) \quad \text{for all } (v,t) \in B^4 \times [-\pi, \pi].$$

3. Explain how C can be made continuous all the way to $\overline{B}^4 \times [-\pi, \pi]$.
4. If Φ is the continuous extension, show that

$$degre(\Phi_{|S^3 \times \{0\}}) = \text{genus of } \Sigma.$$

Part 1

Given $v \in B^4$, consider $F_v : S^3 \to S^3, \quad x \mapsto \frac{1-|v|^2}{|x-v|^2}(x-v) - v$.



Given $\Sigma = \partial A$ close surface in S^3 consider

$$C : B^4 \times [-\pi, \pi] \to \mathcal{Z}_2(S^3), \quad C(v,t) = \partial\{x \in S^3 : d(x, F_v(\Sigma)) < t\},$$

where $d(x, F_v(\Sigma))$ is negative if $x \in F_v(A)$ and positive if $x \in S^3 \setminus \overline{F_v(A)}$.

- $C(v, \pi) = C(v, -\pi) = 0$;
- C is continuous only in the flat topology;

Part 2

Lemma

$area(C(v,t)) \leq \mathcal{W}(\Sigma)$ for all $(v,t) \in B^4 \times [-\pi, \pi]$.

Proof. Fix (v,t) and set

$$\psi : F_v(\Sigma) \to S^3, \quad x \mapsto \cos t\, x + \sin t\, N_v(x),$$

where N_v =exterior unit normal to $F_v(\Sigma)$.

- $$C(v,t) \subset \{Jac\,\psi \geq 0\} \implies area(C(v,t)) \leq \int_{\{Jac\,\psi \geq 0\}} Jac\,\psi$$
- If e_1, e_2 are principal directions at $x \in F_v(\Sigma)$

  $$D\psi(e_i) = \cos t\, e_i - k_i \sin t\, e_i \implies Jac\,\psi = (\cos t - k_1 \sin t)(\cos t - k_2 \sin t)$$

  $$= \cos^2 t + k_1 k_2 \sin^2 t - (k_1 + k_2) \sin t \cos t$$
  $$\leq \cos^2 t + H^2 \sin^2 t - 2H \sin t \cos t$$

  $$= (\cos t - H \sin t)^2 \leq 1 + H^2.$$
- $$area(C(v,t)) \leq \int_{\{Jac\,\psi \geq 0\}} Jac\,\psi \leq \int_{F_v(\Sigma)} 1 + H^2 = W(F_v(\Sigma)) = W(\Sigma)$$

Part 3

The map $C : B^4 \times [-\pi, \pi] \to \mathcal{Z}_2(S^3)$ is *not* continuous on $S^3 \times [-\pi, \pi]$.

1) If $v_i \in B^4$ tends to $p \in \Sigma$ orthogonally and with slope θ then

$$F_{v_i}(\Sigma) \to \partial B_{\frac{\pi}{2}-\theta}(-\cos\theta N(p) - \sin\theta p),$$

$$C(v_i, t) \to \partial B_{\frac{\pi}{2}-\theta+t}(-\cos\theta N(p) - \sin\theta p).$$

2) We find $T : \bar{B}^4 \to \bar{B}^4$ continuous which projects a tubular neighborhood of Σ onto Σ and so that

- $\tilde{C}(v, t) = C(T(v), t)$ extends continuously to $\overline{B}^4 \times [-\pi, \pi]$;
- $\tilde{C}(S^3, t) \subset \{\text{round spheres in } S^3\}$

3) Finally, we find a suitable continuous function h so that

$$\Phi : \bar{B}^4 \times [-\pi, \pi] \to \mathcal{Z}_2(S^3), \quad \Phi(v, t) = \tilde{C}(v, h(v, t))$$

and for all $p \in S^3$

- $\Phi(p, t)$ is a sphere of radius $\pi/2 + t/2$ centered at some $Q(p) \in S^3$.

Part 4

Lemma

The map

$$\Phi : S^3 \times \{0\} \to \{\text{oriented great spheres}\} \approx S^3, \quad \Phi(p,0) = \partial B_{\pi/2}(Q(p))$$

has degree$(\phi_{|S^3\times\{0\}}) = g$ the genus of Σ.

Proof.

1) With $S^3 \approx A \cup (S^3 \setminus A) \cup \Sigma \times [-\pi/2, \pi/2]$, the map Q factors as

$$Q(x) = \begin{cases} x & \text{if } x \in A \\ -x & \text{if } x \in S^3 \setminus A \\ -\cos\theta N(p) - \sin\theta p & \text{if } (p,\theta) \in \Sigma \times [-\pi/2, \pi/2] \end{cases}$$

2) $\int Q^*(dVol_{S^3})$

$$= \int_A Q^*(dVol_{S^3}) + \int_{S^3\setminus A} Q^*(dVol_{S^3}) + \int_{\Sigma\times[-\pi/2,\pi/2]} Q^*(dVol_{S^3})$$

$$= Vol(A) + Vol(S^3 \setminus A) + \int_{\Sigma\times[-\pi/2,\pi/2]} Q^*(dVol_{S^3})$$

$$= Vol(S^3) - \int_{\Sigma\times[-\pi/2,\pi/2]}(\cos^2\theta k_1 k_2 + \sin^2\theta - \sin\theta\cos\theta(k_1 + k_2))$$

$$= Vol(S^3) - \int_{\Sigma\times[-\pi/2,\pi/2]}(\cos^2\theta(K-1) + \sin^2\theta)$$

$$= Vol(S^3) - \int_{-\pi/2}^{\pi/2}\cos^2\theta \int_\Sigma K = Vol(S^3) + 2\pi^2(g-1)$$

$$= gVol(S^3) \implies degree(Q) = g!$$